NEURAL NETWORKS IN DETERMINATION OF DEFUZZIFICATION FUNCTIONALS

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ABSTRACT

Ordered fuzzy numbers as generalization of convex fuzzy numbers are defined together with four algebraic operations. For defuzzification operators, that play the main role when dealing with fuzzy controllers and fuzzy inference systems, new representation formulae are given. Step ordered fuzzy numbers are considered. Approximation method based on forward neural networks is presented for determining defuzzification functionals when training sets of data are given. Results of approximation are given.

FUZZY NUMBERS

Fuzzy numbers (Zadeh, 1965) are very special fuzzy sets defined on the universe of all real numbers **R**. In applications the so-called (L, R)–numbers proposed by Dubois and Prade (Dubois & Prade, 1978) as a restricted class of membership functions, are often in use. In most cases one assumes that membership function of a fuzzy number A satisfies convexity assumptions (Nguyen, 1978). However, even in the case of convex fuzzy numbers (CFN) multiply operations are leading to the large grow of the fuzziness, and depend on the order of operations.
This as well as other drawbacks have forced us to think about some generalization. Number of attempts to introduce non-standard operations on fuzzy numbers has been made (Drewniak, 2001; Klir, 1997; Sanschez, 1984; Wagenknecht, 2001). Our main observation made in (Kosiński et.al., 2002a) was: a kind of quasi-invertibility (or quasi-convexity (Martos, 1975)) of membership functions is crucial. Invertibility of membership functions of convex fuzzy number A makes it possible to define two functions a_1, a_2 on $[0, 1]$ that give lower and upper bounds of each α-cut of the membership function μ_A of the number A

$$A[\alpha] = \{x : \mu_A(x) \geq \alpha\} = [a_1(\alpha), a_2(\alpha)]$$

with $a_1(\alpha) = \mu_A|_{incr}^{-1}(\alpha)$ and $a_2(\alpha) = \mu_A|_{decr}^{-1}(\alpha)$, where $|_{incr}$ and $|_{decr}$ denote the restrictions of the function μ_A to its sub-domains on which is increasing or decreasing, respectively. Both functions $a_1(\alpha), a_2(\alpha)$ were used for the first time by the authors of (Goetschel & Voxman, 1986) in their parametric representation of fuzzy numbers, they also introduced a linear structure to convex fuzzy numbers.

ORDERED FUZZY NUMBERS

In the series of papers (Kosiński et.al., 2002a; Kosiński et. al., 2003b,a) we have introduced and then developed main concepts of the space of ordered fuzzy numbers (OFNs). In our approach the concept of membership functions has been weakened by requiring a mere *membership relation* .
Definition 1. *Pair (f, g) of continuous functions such that $f, g : [0, 1] \rightarrow$ **R** is called ordered fuzzy number A.*

Notice that f and g need not be inverse functions of some membership function. If, however, f is increasing and g – decreasing, both on the unit interval I, and $f \leq g$, then one can attach to this pair a continuous function μ and regard it as a membership function a convex fuzzy number with an extra feature, namely the orientation of the number. This attachment can be done by the formula $f^{-1} = \mu|_{incr}$ and $g^{-1} = \mu|_{decr}$. Notice that pairs (f, g) and (g, f) represent two different ordered fuzzy numbers, unless $f = g$. They differ by their orientations.
Definition 2. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula*

$$f_C(y) = f_A(y) \star f_B(y)\, , g_C(y) = g_A(y) \star g_B(y) \quad (1)$$

where "$\star$" works for "$+$", "$-$", "$\cdot$", and "$\div$", respectively, and where $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero.

Scalar multiplication by real $r \in$ **R** is defined as $r \cdot A = (rf_A, rg_A)$. The subtraction of B is the same as the addition of the opposite of B, and consequently $B - B = 0$, where $0 \in$ **R** is the crisp zero. It means that subtraction is not compatible with the the extension

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principle, if we confine OFNs to CFN. However, the addition operation is compatible, if its components have the same orientations. Notice, however, that addition, as well as subtraction, of two OFNs that are represented by affine functions and possess classical membership functions may lead to result which may not possess its membership functions (in general $f(1)$ needs not be less than $g(1)$).

Relation of partial ordering in the space $\mathcal{R}$ of all OFN, can be introduced by defining the subset of *positive* ordered fuzzy numbers: a number $A = (f, g)$ is not less than zero, and write

$$A \geq 0 \text{ if } f \geq 0,\, g \geq 0 \,, \text{and } A \geq B \text{ if } A - B \geq 0\,. \quad (2)$$

In this way the space $\mathcal{R}$ becomes *a partially ordered ring*. Neutral element of addition in $\mathcal{R}$ is a pair of constant function equal to crisp zero.

Operations introduced in the space $\mathcal{R}$ of all ordered fuzzy numbers (OFN) make it an algebra, which can be equipped with a sup norm $||A|| = \max(\sup_{s\in I}|f_A(s)|, \sup_{s\in I}|g_A(s)|)$ if $A = (f_A, g_A)$. In $\mathcal{R}$ any algebraic equation $A+X = C$ for X, with arbitrarily given fuzzy numbers A and C, can be solved. Moreover, $\mathcal{R}$ becomes a Banach space, isomorphic to a Cartesian product of $C(0, 1)$ - the space of continuous functions on $[0, 1]$. It is also a Banach algebra with unity: the multiplication has a neutral element - the pair of two constant functions equal to one, i.e. the crisp one.

Some interpretations of the concepts of OFN have been given in (Kosiński et.al., 2009a). Fuzzy implications within OFN are presented in (Kosiński et. al., 2009b).

STEP NUMBERS

It is worthwhile to point out that the class of ordered fuzzy numbers (OFNs) represents the whole class of convex fuzzy numbers with continuous membership functions. To include all CFN with piecewise continuous membership functions more general class of functions f and g in Def.1 is needed. This has been already done by the first author who in (Kosiński, 2006) assumed they are functions of bounded variation. The new space is denoted by $\mathcal{R}_{BV}$. Then operations on elements of $\mathcal{R}_{BV}$ are defined in the similar way, the norm, however, will change into the norm of the Cartesian product of the space of functions of bounded variations (BV). Then all convex fuzzy numbers are contained in this new space $\mathcal{R}_{BV}$ of OFN. Notice that functions from BV (Łojasiewicz, 1973) are continuous except for a countable numbers of points.

Important consequence of this generalization is the possibility of introducing the subspace of OFN composed of pairs of step functions. It will be done as follows. If we fix a natural number K and split $[0, 1)$ into $K - 1$ subintervals $[a_i, a_{i+1})\,, i = 1, 2, ..., K$, i.e. $\bigcup_{i=1}^{K-1}[a_i, a_{i+1}) = [0, 1)$, where $0 = a_1 < a_2 < ... < a_K = 1$, we may define step function f of resolution K by putting value $f(s) = u_i \in \mathbf{R}$, for $s \in [a_i, a_{i+1})$, then each such function f can be identified with a K-dimensional vector $f \sim \underline{u} = (u_1, u_2...u_K) \in \mathbf{R}^K$, the K-th value u_K corresponds to $s = 1$, i.e. $f(1) = u_K$. Taking pair of such functions we have an ordered fuzzy number from $\mathcal{R}_{BV}$. Now we introduce

Definition 3. *By* step ordered fuzzy number A *of resolution* K *we mean ordered pair* (f, g) *of functions such that* $f, g : [0, 1]\rightarrow\mathbf{R}$ *are* K*-step function.*

We use $\mathcal{R}_K$ for denotation the set of elements satisfying Def. 3. The set $\mathcal{R}_K \subset \mathcal{R}_{BV}$ has been extensively elaborated by our students in (Gruszczyńska & Krejewska, 2008) and (Kościeński, 2010). We can identify $\mathcal{R}_K$ with the Cartesian product of $\mathbf{R}^K \times \mathbf{R}^K$ since each K-step function is represented by its K values. It is obvious that each element of the space $\mathcal{R}_K$ may be regarded as approximation of elements from $\mathcal{R}_{BV}$, by increasing the number K of steps we are getting the better approximation. The norm of $\mathcal{R}_K$ is assumed to be the Euclidean one of $\mathbf{R}^{2K}$, then we have a inner-product structure for our disposal.

DEFUZZIFICATION FUNCTIONALS

In the course of defuzzification operation in CFN to a membership function a real, crisp number is attached. We know number of defuzzification procedures from the literature (Van Leekwijck & Kerre, 1999). Continuous, linear functionals on $\mathcal{R}$ give the class of defuzzification functionals . Each of them, say ϕ, has the representation by the sum of two Stieltjes integrals with respect to two functions h_1, h_2 of bounded variation,

$$\phi(f, g) = \int_0^1 f(s)dh_1(s) + \int_0^1 g(s)dh_2(s)\,. \quad (3)$$

Notice that if for $h_1(s)$ and $h_2(s)$ we put $\lambda H(s)$ and $(1 - \lambda)H(s)$, respectively, with $0 \leq \lambda \leq 1$ and $H(s)$ as the Heaviside function with the unit jump at $s = 1$, then the defuzzification functional in (3) will lead to the classical MOM – middle of maximum, FOM (first of maximum), LOM (last of maximum) and RCOM (random choice of maximum), with an appropriate choice of λ. For example if for $h_1(s)$ and $h_2(s)$ we put $1/2H(s)$ then the defuzzification functional in (3) will represent the classical MOM – middle of maximum

$$\phi(f, g) = 1/2(f(1) + g(1))\,. \quad (4)$$

New model gives the continuum number of defuzzification operators both linear and nonlinear, which map ordered fuzzy numbers into reals. Nonlinear functional can be defined, see (Kosiński & Wilczyńska-Sztyma, 2010), as an example we have center of gravity defuzzification functional (COG) calculated at OFN (f,g) is

$$\bar{\phi}_G(f,g) = \frac{\int_0^1 \frac{f(s)+g(s)}{2}[f(s)-g(s)]ds}{\int_0^1 [f(s)-g(s)]ds} . \quad (5)$$

If $A = c^\ddagger$ then we put $\bar{\phi}_G(c^\ddagger) = c$. When $\int_0^1 [f(s) - g(s)]ds = 0$ in (5) a correction needs to be introduced. Here by writing $\bar{\phi}(c^\ddagger)$ we understand the action of the functional $\bar{\phi}$ on the crisp number $c^\ddagger$ from **R**, which is represented by the pair of constant functions $(c^\dagger, c^\dagger)$, with $c^\dagger(s) = c\,, s \in [0,1]$.

In our understanding the most general class of continuous defuzzification functionals ϕ should satisfy three conditions:

1. $\phi(c^\ddagger) = c\,,$
2. $\phi(A + c^\ddagger) = \phi(A) + c\,,$
3. $\phi(cA) = c\phi(A)$, for any $c \in \mathbf{R}$ and $A \in \mathcal{R}$.

Here by writing $\phi(c^\ddagger)$ we understand the action of the functional ϕ on the crisp number $c^\ddagger$ from **R**, which is represented, in the case of $\mathcal{R}_K$, by the pair of constant functions $(c^\dagger, c^\dagger)$, with $c^\dagger(i) = c\,, i = 1,2,...,K$. The condition 2. is a *restricted additivity*, since the second component is crisp number. The condition 3. requires from ϕ to be homogeneous of order one, while the condition 1. requires $\int_0^1 dh_1(s) + \int_0^1 dh_2(s) = 1$, in the representation (3).

On the space $\mathcal{R}_K$ a representation formula for a general non-linear defuzzification functional $H : \mathbf{R}^K \times \mathbf{R}^K \to \mathbf{R}$ satisfying the conditions 1.– 3., can be given as a linear composition (Kosiński & Wilczyńska-Sztyma, 2010) of arbitrary homogeneous of order one, continuous function G of $2K-1$ variables, with the 1D identity function, i.e.

$$H(\underline{u}, \underline{v}) = u_1 + G(u_2 - u_1, u_3 - u_1, ..., u_K - u_1, v_1 - u_1, v_2 - u_1, ..., v_K - u_1)\,, \quad (6)$$

with

$$\underline{u} = (u_1, ..., u_K)\,, \underline{v} = (v_1, ..., v_K)\,.$$

Due to the fact that $\mathcal{R}_K$ is isomorphic to $\mathbf{R}^K \times \mathbf{R}^K$ we conclude, from the Riesz theorem and the condition 1. that a general linear defuzzification functional on $\mathcal{R}_K$ has the representation

$$H(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{b} + \underline{v} \cdot \underline{d}\,, \quad (7)$$

with arbitrary $\underline{b}\,, \underline{d} \in \mathbf{R}^K$, such that $\underline{1} \cdot \underline{b} + \underline{1} \cdot \underline{d} = 1$,

where $\cdot$ denotes the inner (scalar) product in $\mathbf{R}^K$ and $\underline{1} = (1,1,...,1) \in \mathbf{R}^K$ is the unit vector in $\mathbf{R}^K$, while the pair $(\underline{1}, \underline{1})$ represents a crisp one in $\mathcal{R}_K$. It means that such functional is represented by the vector $(\underline{b}, \underline{d}) \in \mathbf{R}^{2K}$. Notice that functionals of the type $\phi_j = \underline{e}_j, j = 1,2,...,2K$, where $\underline{e}_j \in \mathbf{R}^{2K}$ has all zero component except for 1 on the j-th position, form a basis of $\mathcal{R}_K{}^*$ - the space adjoint to $\mathcal{R}_K$, they are called *fundamental functionals* .

Notice that each real-valued function $\psi(z)$ of a real variable $z \in \mathbf{R}$ may be transformed to a fuzzy-valued function on $\mathcal{R}_{BV}$, and even simpler on $\mathcal{R}_K$. Here we have used the representation for $\underline{u} = (u_1, ..., u_K)$ and for $\underline{v} = (v_1, ..., v_K)$.

APPROXIMATION OF DEFUZZIFICATION FUNCTIONALS

Ultimate goal of fuzzy logic is to provide foundations for approximate reasoning. It uses imprecise propositions based on a fuzzy set theory developed by L.Zadeh, in a way similar to the classical reasoning using precise propositions based on the classical set theory. Defuzzification is the main operation which appears in fuzzy controllers and fuzzy inference systems where fuzzy rules are present. It was extensively discussed by the authors of (Van Leekwijck & Kerre, 1999). They have classified the most widely used defuzzification techniques into different groups, and examined the prototypes of each group with respect to the defuzzification criteria.

The problem arises when membership functions are not continuous or do not exist at all. Here on particular subsets of fuzzy sets, namely *step ordered fuzzy numbers* approximation formula of a defuzzification functionals will be searched based on some number of training data. This is a quite new problem never investigated within step ordered fuzzy numbers.

Problem formulation Let finite set of training data be given in the form of N pairs: ordered fuzzy number and value (of action) of a defuzzification functional on it, i.e. $\mathsf{TRE} = \{(A_1, r_1), (A_2, r_2), ..., (A_N, r_N)\}$. For a given small ϵ find a continuous functional $H : \mathcal{R}_K \to \mathbf{R}$ which approximates the values of the set TRE within the error smaller than ϵ, i.e. $\max_{1 \le p \le N} |H(A_p) - r_p| \le \epsilon$, where $(A_p, r_p) \in \mathsf{TRE}$.

Problem may possess several solution methods, e.g. a dedicated evolutionary algorithm (Kosiński, 2007; Kosiński & Markowska-Kaczmar, 2007) or an artificial neural network. We have use the representation (6) of the searched defuzzification functional in which a homogeneous, of order one, function Ψ appears. It means that values of this function are determined from its arguments situated on the unit sphere $\mathcal{S}_{2K-1}$ in $2K-1$ D space. Here an artificial neural network will be in use.

Training and test sets used in the further section (from now denoted as TRE and TES, respectively) have the following form. Set of N elements is composed of N pairs of OFN and value of a defuzzification functional on it, i.e.: $\{(A_1, r_1), (A_2, r_2), ..., (A_N, r_N)\}$. We are training an artificial neural network on this set to find the approximated form of the functional.

NEURAL NETWORK SIMULATIONS

In order to make approximation of linear and the non-linear defuzzification functionals on step ordered fuzzy numbers (SOFN) a package of artificial neural networks (ANN) has been used. Since each SOFN is represented by a vector of $2K$ number, each input to artificial neural networks has $2K$ real-valued components. In our case it was:

- MLP neural network with one hidden layer:
  - 20 inputs
  - 5 neurons in hidden layers
  - 1 output neuron
- 500 iterations
- data set:
  - training - TRE_0, TRE_4
  - testing - TES_0, TES_4

Data generation

The procedure to generate TRE and TES sets was the following.

1. Generate 60 random points on a $2K-1$ dimensional hyper-sphere, where $K = 10$. Let $\varphi = (u_2, u_3, ..., u_{K-1}, v_1, v_2, ..., v_K)$ be one of these points. All points fulfill the conditions $u_n < u_{n+1}$ and $v_m > v_{m+1}$. This ensures that the generated fuzzy numbers have a trapezoidal shape. In the further parts this assumption has been omitted.

2. Generate two sets of fuzzy numbers using the following methods of generating a value of u
   - $u = 0$
   - u is a random value from $(-4, 4)$

3. For each fuzzy number find the defuzzified value and split the sets in ratio 2:1 to form:
   - TRE_0 and TES_0 from fuzzy numbers with $u_1 = 0$
   - TRE_4 and TES_4 from fuzzy numbers with $u_1 \in (-4, 4)$

The general strategy was to train the network with data sets having $2K$ inputs and an output representing the discrete values of fuzzy output values and the crisp output calculated according to selected standard defuzzification algorithms. For the linear defuzzification we have used: MOM (middle of maximum), LOM (last of maximum) and FOM (first of maximum).

Table 1 presents the final training MSE (for RSME[%]) for all the used methods. Table 2 presents

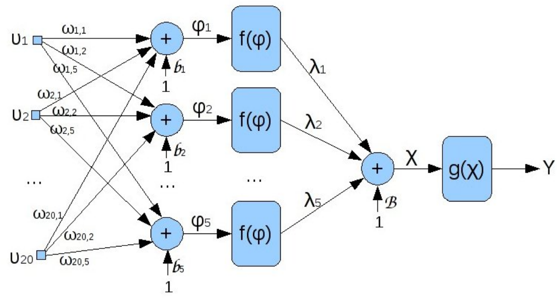

Figure 1: Neural network structure

the final training gradient for all the used methods. The error was calculated as

$$Error^2 = \frac{1}{P}\sum_{i=1}^{i=P}(H(A_i) - r_i)^2 .$$

We have used the Lavenberg-Marquardt adaptation algorithm.

Training Set	MOM	LOM	FOM
TRE_0	1.196156E-11	1.17966E-11	3.2052E-11
TRE_4	8.22773E-10	1.51997E-9	3.1339E-9

Table 1: Final training RMSE

Training Set	MOM	LOM	FOM
TRE_0	3.57907E-6	1.14851E-6	1.09344E-6
TRE_4	0.001232	0.0001864	0.00311

Table 2: Final training gradient

The validation of our neural network is done by testing the network with TES_0 and TES_4 data sets generated with all of the following defuzzification methods : MOM(middle of maximum), LOM (last of maximum) and FOM (first of maximum).

The validation of data TES_0 and TES_4 defuzzified with MOM strategy converges successfully. The results are presented at the figures: for TRE_0 MSE [%] (Figure 2), gradient (Figure 3), for TRE_4 RMSE (Figure 4), gradient (Figure 5). Similar results have been obtained for other defuzzification methods. On the X axis we have we have number of iterations. In histograms the number of sample appears.

Performed simulation proved that ANN can successfully represent the defuzzification strategies. Linear approximations of defuzzification functionals with MOM, LOM and FOM were correct. The trained ANN approximations for all the methods were successfully tested with TES_0 and TES_4 data sets. Table 3 presents the final validation RMSE for all the used methods.

Testing Set	MOM	LOM	FOM
TES_0	1.781138E-5	3.020065E-5	0.0001056
TES_4	4.300E-9	2.02054E-6	0.0006829

Table 3: Final linear validating RMSE[%]

NONLINEAR DEFUZZIFICATION FUNCTIONAL

Similar method has been used for nonlinear defuzzification functional, namely for the center of gravity (COG). The validation of data TES_0 and TES_4 defuzzified with COG strategy converges successfully.

Transfer functions

The first layer transfer function is given by the formula:

$$f(x) = \frac{2}{1+e^{-2x}} - 1$$

The hidden layer transfer function is given by $g(x) = x$, and the output is given by

$$Y = g(X) = X = \sum_{j=1}^{5} \Phi_j \lambda_j + B$$

where $\Phi_j = f(\varphi_j) = f(\sum_{i=1}^{20} u_i * \omega_{i},_j + b_j)$. Hence we have

$$Y = \sum_{j=1}^{5} f(\sum_{i=1}^{20} u_i \omega_{i},_j + b_j))\lambda_j + B \,.$$

The weights and other parameters can be listed in the form of tables. Due to the lack of the space it is not presented here.

CONCLUSION

The present paper brings an outline of the results of approximation of defuzzification functional of step ordered fuzzy number that have been obtained with a help of the tool of the computational intelligence, namely of artificial neural networks. We can conclude that the tool is helpful. It is rather evident that further research in this field should follow.

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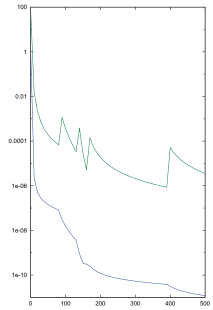

Figure 2: MOM on TRE_0: MSE [%] and Gradient

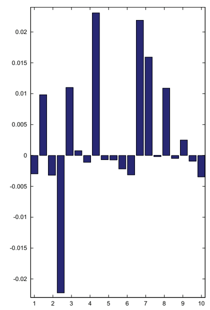

Figure 3: MOM on TES_0: MSE [%]

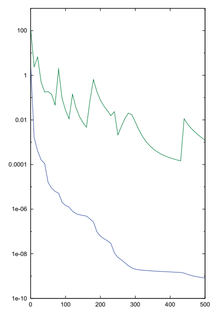

Figure 4: MOM on TES_4: MSE [%] and Gradient

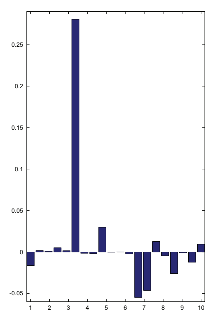

Figure 5: MOM on TES_4: RMSE

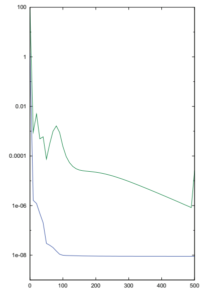

Figure 6: COG on TRE_0: RMSE and gradient

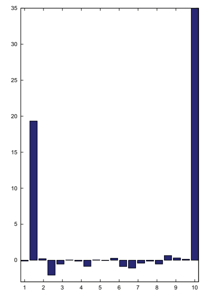

Figure 7: COG on TES_0: RMSE

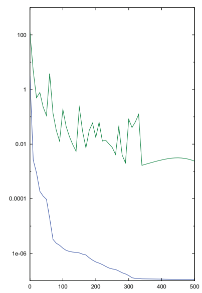

Figure 8: COG on TES_4: RMSE and gradient

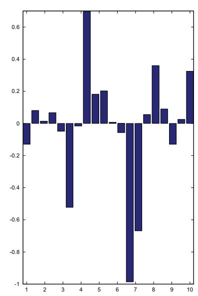

Figure 9: COG on TES_4: RMSE