VOLATILITY SURFACE CALIBRATION IN ILLIQUID MARKET ENVIRONMENT

László Nagy
Mihály Ormos
Department of Finance
Budapest University of Technology and Economics
Magyar tudósok körútja 2, Budapest H-1117, Hungary
E-mail: nagyl@finance.bme.hu, ormos@finance.bme.hu

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ABSTRACT

In this paper, we show the fragility of widely-used Stochastic Volatility Inspired (SVI) methodology. Especially, we highlight the sensitivity of SVI to the fitting penalty function. We compare different weight functions and propose to use a novel methodology, the implied vega weights. Moreover, we unveil the relationship between vega weights and the minimization task of observed and fitted price differences. Besides, we show that implied vega weights can stabilize SVI surfaces in illiquid market conditions.

INTRODUCTION

Vanilla options are traded with finite number of strikes and maturities. Thus, we can observe only some points of the implied volatility surface. It is known that vanilla prices are arbitrage free hence exotic option traders would like to calibrate their prices to vanillas (Dupire 1994). The main difficulty is that calibration methods need the implied volatility surface. To overcome this problem we have to construct an arbitrage free surface from the observed points (Schönbucher 1998, Gatheral 2013). In this paper we provide a robust arbitrage free surface fitting methodology.
Chapters are structured as follows: Section 2. is a brief overview of SVI. In Section 3. we compare the different weight functions and present our implied vega weight L^1 methodology. In Section 4. we summarize the findings.

SVI

After the Black Monday in 1987, traders behavior changed. Implied volatility skew became more pronounced. Risk aversion incorporated in the volatility. Hence, risk transfers between tenors and strikes get more sophisticated.
The changes were in line with human nature, because people have different risk appetite in different tenors. Moreover, extreme high out of money implied volatilities are consequences of risk aversion and fear of the unpredictable.

Besides, our risk neutral risk assessment should be consequent. Otherwise, calendar and butterfly arbitrage opportunities appear;

$$C(K, \tau_1) < C(K, \tau_2) \text{ if and only if } \tau_1 < \tau_2 \quad (1)$$

$$C(K_1, \tau) - \frac{K_3 - K_1}{K_3 - K_2} C(K_2, \tau) + \frac{K_2 - K_1}{K_3 - K_2} C(K_3, \tau) > 0 \quad (2)$$

where $C(K, \tau)$ represents the price of a European call option with strike K and maturity τ.

Considering the behavior of compound interest the Black-Scholes log-normal model is applicable.

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (3)$$

Also we have seen the volatility surface is not flat hence this model needs some adjustments. The most straightforward correction leads to the Local Volatility model (Dupire 1994).

$$dS_t = rS_t dt + \sigma(S_t, t) S_t dW_t \quad (4)$$

However, calculating implied and realized volatilities show that Local Volatility is only an idealized perfect fit, because volatility is stochastic.

$$dS_t = rS_t dt + \sigma_t^{Spot} S_t dW_t \quad (5)$$
$$d\sigma_t^{BS} = u(k,t)dt + \gamma(k,t)dW_t + \sum_{i=1}^{n} v_i(k,t)\, dW_t^i$$

where $W, W_1, \ldots W_n$ are independent Brownian motions, k is the log-moneyness and σ^{BS} denotes the Black-Scholes implied volatility.

Schönbucher showed that the spot volatility can not be an arbitrary function of implied volatility, because of the static arbitrage constraints.

$$\sigma^{Spot} = \frac{-\gamma k}{\sigma^{BS}(k,T)} \pm \sqrt{\sigma^{BS}(k,T) + \frac{k^2}{\left(\sigma^{BS}(k,T)\right)^2}\left(\sum_{i=1}^{n} v_i^2 - \gamma^2\right)} \quad (6)$$

Besides the arbitrage constraints, Heston's model sheds more light on implied volatility modeling. Gatheral et

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al. proposed the so called SVI (Stochastic Volatility Inspired) function to estimate all the implied volatility surface;

$$\sigma^{BS} = \mathrm{a} + \mathrm{b}\left(\rho(\mathrm{k}-\mathrm{m}) + \sqrt{(\mathrm{k}-\mathrm{m})^2 + \sigma^2}\right) \quad (7)$$

where a controls the level, b the slopes of the wings, ρ the counter-clockwise rotation, m the location and σ the at the money curvature of the smile. The SVI model has compelling fitting results, in addition it implies a static arbitrage free volatility surface. The only arguable step in the methodology is the model calibration. Kos et al. (2013) proposed to minimize the square differences between observed and fitted volatility, while Homescu (2011) advised a square difference method. Nevertheless West (2005) applied vega weighted square volatility differences. Zelida system (2009) used total implied variances. An other noticeable approach comes from Gatheral et al. (2013) who minimized squared price differences, but there are further regression based models as well (Romo 2011).
In this article we propose a new absolute price difference based approach to stabilize SVI in illiquid market conditions.

SENSITICITY ANALAYSIS OF SVI

At the money options are more liquid than far out of money options, hence the bid-ask spread widens along the wings. This implies larger price ambiguity of OTM option prices. In order to stabilize the implied volatility surface we have to penalize price ambiguity.

Uniform weights

Highlighting the problem we can apply uniform weights. This approach assumes that all of the information is equally relevant. Thus, we get the usual square distance optimization task (Zelida 2009, Kos 2013);

$$\min_{\sigma^{\mathrm{Fit}}_{\mathrm{K},\tau} \in C^0} \sum_{\tau\in\mathcal{T}} \sum_{K\in\mathcal{K}} \left(\sigma^{\mathrm{Fit}}_{\mathrm{K},\tau} - \sigma^{\mathrm{BS}}_{\mathrm{K},\tau} \right)^2 \quad (8)$$

where $\mathcal{K}$ and $\mathcal{T}$ represent the sets of the traded strikes and maturities, $\sigma^{\mathrm{BS}}_{\mathrm{K},\tau}$ is the implied and $\sigma^{\mathrm{Fit}}_{\mathrm{K},\tau}$ is the fitted volatility.

1. Note that for fixed maturity $\sigma^{\mathrm{BS}}_{\mathrm{K,T}}$ increases in $|\mathrm{K} - \mathrm{F_T}|$. The volatility bid-ask spread also widens along the wings. This implies that $\sigma^{\mathrm{BS,Ask}}_{\mathrm{K,T}}$ increases faster than $\sigma^{\mathrm{BS,Bid}}_{\mathrm{K,T}}$hence defining the fair value of a deep OTM option from bid and ask price is not straightforward.

2. Furthermore, deep out of money implied volatilities are usually higher than ATM volatilities. Therefore, uniform square penalty overfits the wings and underfits the ATM range.

3. In addition, the set of traded strikes is not stable in time. Therefore, the estimated surface will be unstable in time.

Data truncation

The simplest approach to solve the problem could be just using close ATM prices to fit SVI and then extrapolate along wings. The main drawback of this method is that OTM short dated options contain the market anticipated tail risk information. Truncating the data stabilizes the surface and implies accurate long term fit, but underestimates tail risk hence underprices exotic products.

Square of price differences

The most popular optimization technique is minimizing L^2 distances. The main drawback of this approach is fitting to the mean, instead of the median which implies outlier sensitivity.

Vega weights

In order to deal with the skew and price ambiguity we propose to use a natural Gaussian based weight function. It turned out that truncating the data do not give the appropriate results. Therefore, we have to find a weight function which minimizes L^1 distance, penalizes price ambiguity, but still able to use tail risk information.

$$\min_{\sigma^{\mathrm{Fit}}_{\mathrm{K},\tau} \in C^0} \sum_{\tau\in\mathcal{T}} \sum_{K\in\mathcal{K}} \mathrm{w}(\mathrm{K},\tau) \mid \sigma^{\mathrm{Fit}}_{\mathrm{K},\tau} - \sigma^{\mathrm{BS}}_{\mathrm{K},\tau} \mid \quad (9)$$

Note that the above optimization problem is still not general enough, because weight is a function of strike and maturity. This incorporates a sticky strike assumption. However, if we add the spot price S_0 as another independent variable to the weight, then we can get more general penalty functions.

$$\min_{\sigma^{\mathrm{Fit}}_{\mathrm{K},\mathrm{S_0},\tau} \in C^0} \sum_{\tau\in\mathcal{T}} \sum_{K\in\mathcal{K}} \mathrm{w}(\mathrm{K},\mathrm{S_0},\tau) \mid \sigma^{\mathrm{Fit}}_{\mathrm{K},\mathrm{S_0},\tau} - \sigma^{\mathrm{BS}}_{\mathrm{K},\mathrm{S_0},\tau} \mid (10)$$

Our initial problem is to find an implied volatility surface. This means that we would like to penalize observed and fitted volatility differences.
Practitioners need the surface for trading. Hence, they are interested in the dollar amount of the discrepancies between fitted and observed volatilities.

$$\min_{C^{Fit}_{K,S_0,\tau} \in C^0} \sum_{\tau\in\mathcal{T}} \sum_{K\in\mathcal{K}} |C^{Fit}_{K,S_0,\tau} - C^{BS}_{K,S_0,\tau}| \quad (11)$$

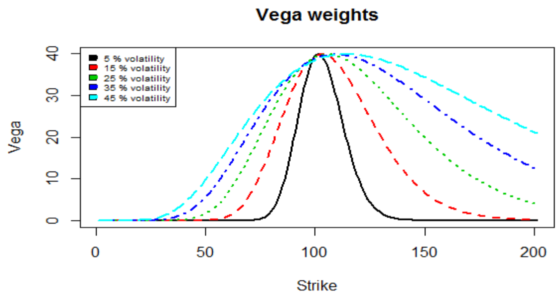
After some calculations in Appendix, we can see that optimizing the price differences is approximately the same as optimizing vega weighted implied volatility differences.

$$\min_{\sigma^{\mathrm{Fit}}_{\mathrm{K},\mathrm{S_0},\tau} \in C^0} \sum_{\tau\in\mathcal{T}} \sum_{K\in\mathcal{K}} \mathcal{V}^{BS}_{\mathrm{K},\mathrm{S_0},\tau} \mid \sigma^{\mathrm{Fit}}_{\mathrm{K},\mathrm{S_0},\tau} - \sigma^{\mathrm{BS}}_{\mathrm{K},\mathrm{S_0},\tau} \mid \quad (12)$$

Using the definition of $\mathcal{V}^{BS}_{K,S_0,\tau}$ we get the price difference implied weight function.

$$w(K, S_0, \tau) = S_0 e^{-q\tau} \varphi(d_1)\sqrt{\tau} \tag{13}$$

where $\varphi(x)$ represents the standard normal distribution function, d_1 is the standard notation from the Black-Scholes formula and q is the continuous dividends rate.



Figures 1: Vega weights as function of strike, and volatility $\sigma \in [5,15,\dots,45], S_0 = 100$, $K \in [0,\dots,200]$ and $T = 1$ year

Supposing that q, r, σ, S_0, τ are fixed and using the definition of d_1 we get that the weight function has a Gaussian shape in log moneyness $K = Fe^k$.

$$w(K) = S_0\, e^{-q\tau}\, \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{\ln\frac{F}{K}+\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right)^2} = \mathcal{O}\left(e^{-k^2}\right) \tag{14}$$

This implies that we highly penalize fitting discrepancies in the ATM range, while we are lenient with deep OTM fits.
The next step is to fix q, e, S_0, τ and use the first order Taylor approximation of $\sigma(K, S_0, \tau)$ around ATM log moneyness.

$$w\big(\sigma(k)\big) = \frac{S_0 e^{-q\tau}}{2\pi} e^{-\frac{1}{2}\left(\frac{-2k+\left(\sigma(0,S_0,\tau)+\Psi(S_0,\tau)k+\mathcal{O}(k^2)\right)^2\tau}{2\left(\sigma(0,S_0,\tau)+\Psi(S_0,\tau)k+\mathcal{O}(k^2)\right)\sqrt{\tau}}\right)^2} \sqrt{\tau} \tag{15}$$

Hence we get;

$$w\big(\sigma(k)\big) \approx \mathcal{O}(e^{-k^4}) \tag{16}$$

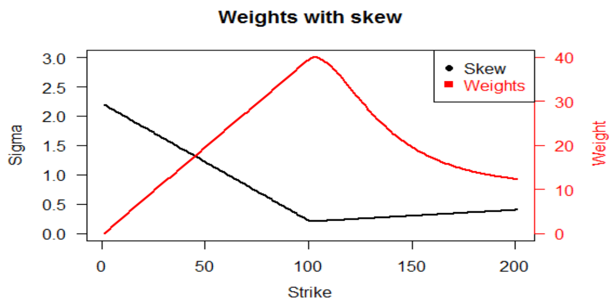
ATM skew is represented by $\Psi(S_0, \tau)$. The asymptotic behavior of $w(\sigma(k))$ shows that the vega weighted implied volatility surface would be stable against extreme OTM implied volatilities.
Moreover, it also can be seen that if k is close to zero then for implied volatility skew and smile we get rather flat vega weights.

$$-2k + \left(\sigma(0, S_0, \tau) + \Psi(S_0, \tau)k + \mathcal{O}(k^2)\right)^2 = \\ -2k\left(1 - \sigma(0, S_0, \tau)\Psi(S_0, \tau)\right) + \sigma(0, S_0, \tau)^2 + \mathcal{O}(k^2) \tag{17}$$

Dividing with $\sigma(0, S_0, \tau)$ we get:

$$\frac{-2k\left(1 - \sigma(0, S_0, \tau)\Psi(S_0, \tau)\right) + \sigma(0, S_0, \tau)^2 + \mathcal{O}(k^2)}{2\left(\sigma(0, S_0, \tau) + \Psi(S_0, \tau)k + \mathcal{O}(k^2)\right)}$$

This function is rather constant if k is small. Equation 14. also shows that for bigger $|k|$ values the weight should decrease with approximately $\exp(-k^2)$.



Figures 2: Implied weights as function of strike, parameters: $S_0 = 100, \sigma_{0,100,1} = 0.2, K \in [0,\dots,200]$, slopes $= (2\%, 0.2\%)$

Figure 2. unveils that vega weights take into account wings, but the bigger the $|k|$ the larger the impact of the $\exp(-k^2)$ term which balances the increasing OTM volatility. Hene, vega weights provide a balanced SVI fit.

Empirical test

In order to lend more color to the fragility of fitting methodology we simulated illiquid market environment by picking 5 data points in each slice from SPX 15/09/2015 option data set (Gatheral 2013). To highlight the outlier-sensitivity we stressed the volatility of the last tenor (T=1.75), moneyness k=0.2 point by 10%.

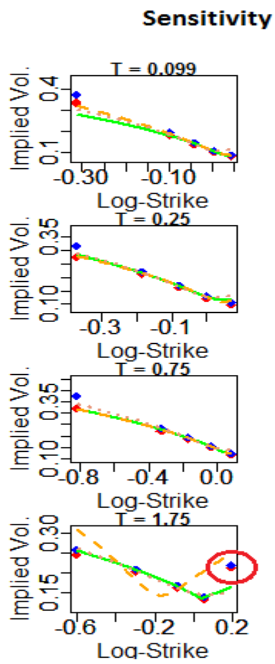


Figure 3: fitted SVI surfaces to filtered SPX 15/09/2015 data, solid lines: price difference based fit (implied vega weights), dashed lines: square price difference based fit, dotted line: square volatility difference based fit

The results show that if we use L^2 optimization techniques then we overpenalize outliers. It also can be seen using absolute difference based optimization makes the SVI fit stable. In illiquid market environment it is crucial because using square difference based fits only one outlier could have a huge impact on the affected slice or even the all surface, thus destabilizing option prices.

CONCLUSION

We showed that the absolute price difference based SVI fitting methodology is able to stabilize the implied volatility surface. Moreover, we shed some lights on the asymptotic behavior of the weights and displayed the connection with vega weights. We also stressed that absolute price difference based optimization do not assume any specific stickiness, hence it can be used in every volatility regime.

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László Nagy

László Nagy is a PhD student at the Department of Finance, Institute of Business at the School of Economic and Social Sciences, Budapest University of Technology and Economics. His main area of research is financial risk measures and asset pricing. Laszló earned his BSc and MSc in Mathematics with major of financial mathematics at the School of Natural Sciences at Budapest University of Technology and Economics. He is teaching investments and working on his PhD thesis. Before his PhD studies he worked at Morgan Stanley on risk modeling.

Mihály Ormos

Mihály Ormos is a Professor of Finance at the Department of Finance, Institute of Business at the School of Economic and Social Sciences, Budapest University of Technology and Economics. His area of research is financial economics especially asset pricing, risk measures, risk perception and behavioral finance. He serves as one of the contributing editors at Eastern European Economics published by Taylor and Francis. His teaching activities concentrate on financial economics, investments and accounting. Prof. Ormos published his research results in Journal of Banking and Finance, Quantitative Finance, Finance Research Letters, Economic Modelling, Empirica, Eastern European Economics, Baltic Journal of Economics, PLoS One, Acta Oeconomica and Economic Systems amongst others.

APPENDIX

$$\min_{C^{Fit}_{K,S_0,\tau}\in C^0} \sum_{\tau\in\mathcal{T}}\sum_{K\in\mathcal{K}} \left|C^{Fit}_{K,S_0,\tau} - C^{BS}_{K,S_0,\tau}\right|$$

$$= \min_{C^{Fit}_{K,S_0,\tau}\in C^0} \sum_{\tau\in\mathcal{T}}\sum_{K\in\mathcal{K}} \left|S_0 e^{-q\tau}\left(\Phi^{SVI}_{(d_1)} - \Phi^{BS}_{(d_1)}\right) - e^{r\tau}K\left(\Phi^{SVI}_{(d_2)} - \Phi^{BS}_{(d_2)}\right)\right|$$

$$
\begin{aligned}
&= \min_{C^{Fit}_{K,S_0,\tau} \in \mathrm{C}^0} \sum_{\tau \in \mathcal{T}} \sum_{K \in \mathcal{K}} \left| \left(S_0 e^{-q\tau} \varphi(d_1^{BS}) \frac{\ln \frac{K}{F_\tau} + \frac{\sigma^{BS}\sigma^{SVI}}{2}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right.\right. \\
&\left. - K e^{-r\tau} \varphi(d_2^{BS}) \frac{\ln \frac{K}{F_\tau} - \frac{\sigma^{BS}\sigma^{SVI}}{2}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right) (\sigma^{SVI} - \sigma^{BS}) \\
&\left. + \mathcal{O}\left(\left(\frac{\sigma^{SVI} - \sigma^{BS}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right)^2 \right) \right| \\
&= \min_{C^{Fit}_{K,S_0,\tau} \in \mathrm{C}^0} \sum_{\tau \in \mathcal{T}} \sum_{K \in \mathcal{K}} \left| \left(\mathcal{V}^{BS}_{K,\tau} \frac{\ln \frac{K}{F_\tau} + \frac{\sigma^{BS}\sigma^{SVI}}{2}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right.\right. \\
&\qquad \left. - \mathcal{V}^{BS}_{K,\tau} \frac{\ln \frac{K}{F_\tau} - \frac{\sigma^{BS}\sigma^{SVI}}{2}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right) (\sigma^{SVI} \\
&\qquad \left. - \sigma^{BS}) + \mathcal{O}\left(\left(\frac{\sigma^{SVI} - \sigma^{BS}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right)^2 \right) \right| \\
&= \min_{C^{Fit}_{K,S_0,\tau} \in \mathrm{C}^0} \sum_{\tau \in \mathcal{T}} \sum_{K \in \mathcal{K}} \left| \mathcal{V}^{BS}_{K,\tau} (\sigma^{SVI} - \sigma^{BS}) \right. \\
&\qquad \left. + \mathcal{O}\left(\left(\frac{\sigma^{SVI} - \sigma^{BS}}{\sigma^{BS}\sigma^{SVI}\sqrt{\tau}} \right)^2 \right) \right| \\
&\approx \min_{C^{Fit}_{K,S_0,\tau} \in \mathrm{C}^0} \sum_{\tau \in \mathcal{T}} \sum_{K \in \mathcal{K}} \left| \mathcal{V}^{BS}_{K,\tau} (\sigma^{SVI} - \sigma^{BS}) \right|
\end{aligned}
$$

Note that $\mathcal{V}\ is\ o(\sqrt{\tau})$, thus options with short expiry are not vega sensitive.