GENERALIZED GAMMA DISTRIBUTIONS AS MIXED EXPONENTIAL LAWS AND RELATED LIMIT THEOREMS

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ABSTRACT

A theorem due to L. J. Gleser stating that a gamma distribution with shape parameter no greater than one is a mixed exponential distribution is extended to generalized gamma distributions introduced by E. W. Stacy as a special family of lifetime distributions containing both gamma distributions, exponential power and Weibull distributions. It is shown that the mixing distribution is a scale mixture of strictly stable laws concentrated on the nonnegative halfline. As a corollary, the representation is obtained for the mixed Poisson distribution with the generalized gamma mixing law as a mixed geometric distribution. Limit theorems are proved establishing the convergence of the distributions of statistics constructed from samples with random sizes obeying the mixed Poisson distribution with the generalized gamma mixing law including random sums to special normal mixtures.

INTRODUCTION

A. *Motivation*

In most papers dealing with the statistical analysis of meteorological data available to the authors, the suggested analytical models for the observed statistical regularities in precipitation are rather ideal and far from being adequate. For example, it is traditionally assumed that the duration of a wet period (the number of subsequent wet days) follows the geometric distribution (for example, see [1]). Perhaps, this prejudice is based on the conventional interpretation of the geometric distribution in terms of the Bernoulli trials as the distribution of the number of subsequent wet days ("successes") till the first dry day ("failure"). But the framework of Bernoulli trials assumes that the trials are independent whereas a thorough statistical analysis of precipitation data registered in different points demonstrates that the sequence of dry and wet days is not only independent, but it is also devoid of the Markov property so that the framework of Bernoulli trials is absolutely inadequate for analyzing meteorological data.

It turned out that the statistical regularities of the number of subsequent wet days can be very reliably modeled by the negative binomial distribution with the shape parameter less than one. For example, in [2] the data registered in so climatically different points as Potsdam (Brandenburg, Germany) and Elista (Kalmykia, Russia) was analyzed and it was demonstrated that the fluctuations of the numbers of successive wet days with very high confidence fit the negative binomial distribution with shape parameter $r \approx 0.8$. In the same paper a schematic attempt was undertaken to explain this phenomenon by the fact that negative binomial distributions can be represented as mixed Poisson laws with mixing gamma-distributions whereas the Poisson distribution is the best model for the discrete stochastic chaos (see, e. g., [3], [4])

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and the mixing distribution accumulates the stochastic influence of factors that can be assumed exogenous with respect to the local system under consideration.

In the present paper we try to give further theoretic explanation of the high adequacy of the negative binomial model. For this purpose we use the concept of a mixed geometric law introduced in [5] (also see [6], [7]). Having first proved that any generalized gamma distribution (*GG-distribution*) with shape parameter less than one is mixed exponential and thus generalizing Gleser's similar theorem on gamma-distributions [8], we then prove that any mixed Poisson distribution with the generalized gamma mixing law (*GG-mixed Poisson distribution*) is actually mixed geometric. The mixed geometric distribution can be interpreted in terms of the Bernoulli trials as follows. First, as a result of some "preliminary" experiment the value of some random variables taking values in $[0,1]$ is determined which is then used as the probability of success in the sequence of Bernoulli trials in which the original "unconditional" mixed Poisson random variable is nothing else than the "conditionally" geometrically distributed random variable having the sense of the number of trials up to the first failure. This makes it possible to assume that the sequence of wet/dry days is not independent, but is conditionally independent and the random probability of success is determined by some outer stochastic factors. As such, we can consider the seasonality or the type of the cause of a rainy period.

The obtained results can serve as a theoretical explanation of some mixed models used within the popular Bayesian approach to the statistical analysis of lifetime data related to high performance information systems.

B. *Notation and definitions*

In the paper, conventional notation is used. The symbols $\stackrel{d}{=}$ and $\Longrightarrow$ denote the coincidence of distributions and convergence in distribution, respectively. The integer and fractional parts of a number z will be respectively denoted $[z]$ and $\{z\}$.

In what follows, for brevity and convenience, the results will be presented in terms of random variables (r.v:s) with the corresponding distributions. It will be assumed that all the r.v:s are defined on the same probability space $(\Omega,\,\mathfrak{F},\,\mathsf{P})$.

A r.v. having the gamma distribution with shape parameter $r>0$ and scale parameter $\lambda>0$ will be denoted $G_{r,\lambda}$,

$$\mathsf{P}(G_{r,\lambda}<x)=\int_0^x g(z;r,\lambda)dz,$$

with

$$g(x;r,\lambda)=\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x},\ x\geq 0,$$

where $\Gamma(r)$ is Euler's gamma-function, $\Gamma(r)=\int_0^\infty x^{r-1}e^{-x}dx$, $r>0$.

In these notation, obviously, $G_{1,1}$ is a r.v. with the standard exponential distribution: $\mathsf{P}(G_{1,1}<x)=\big[1-e^{-x}\big]\mathbf{1}(x\geq 0)$ (here and in what follows $\mathbf{1}(A)$ is the indicator function of a set A).

The gamma distribution is a particular representative of the class of generalized gamma distributions (GG-distributions), which were first described in [9] as a special family of lifetime distributions containing both gamma distributions and Weibull distributions.

DEFINITION 1. A *generalized gamma distribution* (*GG-distribution*) is the absolutely continuous distribution defined by the density

$$g^*(x;r,\gamma,\lambda)=\frac{|\gamma|\lambda^r}{\Gamma(r)}x^{\gamma r-1}e^{-\lambda x^\gamma},\qquad x\geq 0,$$

with $\gamma\in\mathbb{R}$, $\lambda>0$, $r>0$.

The properties of GG-distributions are described in [9], [10]. In what follows we will be interested only in GG-distributions with $\gamma\in(0,1]$. A r.v. with the density $g^*(x;r,\gamma,\lambda)$ will be denoted $G^*_{r,\gamma,\lambda}$.

For a r.v. with the Weibull distribution, a particular case of GG-distributions corresponding to the density $g^*(x;1,\gamma,1)$ and the distribution function (d.f.) $\big[1-e^{-x^\gamma}\big]\mathbf{1}(x\geq 0)$, we will use a special notation W_γ. Thus, $G_{1,1}\stackrel{d}{=}W_1$. It is easy to see that $W_1^{1/\gamma}\stackrel{d}{=}W_\gamma$.

A r.v. with the standard normal d.f. $\Phi(x)$ will be denoted X,

$$\mathsf{P}(X<x)=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-z^2/2}dz,\qquad x\in\mathbb{R}.$$

A r.v. having the Laplace distribution corresponding to the density $f^\Lambda(x)=\frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$, will be denoted Λ.

The d.f. and the density of a strictly stable distribution with the characteristic exponent α and shape parameter θ defined by the characteristic function (ch.f.)

$$\mathfrak{f}_{\alpha,\theta}(t)=\exp\big\{-|t|^\alpha\exp\{-\tfrac{1}{2}i\pi\theta\alpha\mathrm{sign}t\}\big\},\qquad t\in\mathbb{R},$$

wheter $0<\alpha\leq 2$, $|\theta|\leq\min\{1,\frac{2}{\alpha}-1\}$, will be respectively denoted $F_{\alpha,\theta}(x)$ and $f_{\alpha,\theta}(x)$ (see, e. g., [11]). A r.v. with the d.f. $F_{\alpha,\theta}(x)$ will be denoted $S_{\alpha,\theta}$. To symmetric strictly stable distributions there correspond the value $\theta=0$ and the ch.f. $\mathfrak{f}_{\alpha,0}(t)=e^{-|t|^\alpha}$, $t\in\mathbb{R}$. Hence, it is easy to see that $S_{2,0}\stackrel{d}{=}\sqrt{2}X$.

To one-sided strictly stable distributions concentrated on the nonnegative halfline there correspond the values $\theta = 1$ and $0 < \alpha \le 1$. The pairs $\alpha = 1$, $\theta = \pm 1$ correspond to the distributions degenerate in ± 1, respectively. All the rest strictly stable distributions are absolutely continuous. Stable densities cannot be explicitly represented via elementary functions with four exceptions: the normal distribution ($\alpha = 2$, $\theta = 0$), the Cauchy distribution ($\alpha = 1$, $\theta = 0$), the Lévy distribution ($\alpha = \frac{1}{2}$, $\theta = 1$) and the distribution symmetric to the Lévy law ($\alpha = \frac{1}{2}$, $\theta = -1$).

According to the ¡¡multiplication theorem¿¿ (see, e. g., theorem 3.3.1 in [11]) for any admissible pair of parameters (α, θ) and any $\alpha' \in (0, 1]$ the multiplicative representation $S_{\alpha\alpha',\theta} \stackrel{d}{=} S_{\alpha,\theta} \cdot S_{\alpha',1}^{1/\alpha}$ holds, in which the multipliers on the right-hand side are independent. In particular, for any $\alpha \in (0, 2]$

$$S_{\alpha,0} \stackrel{d}{=} X\sqrt{2S_{\alpha/2,1}}, \tag{1}$$

that is, any symmetric strictly stable distribution is a normal scale mixture.

Let $p \in (0, 1)$. By V_p we denote a r.v. having the *geometric distribution* with parameter p: $\mathsf{P}(V_p = k) = p(1-p)^k$, $k = 0, 1, 2, ...$ This means that for any $m \in \mathbb{N}$

$$\mathsf{P}(V_p \ge m) = \sum\nolimits_{k=m}^{\infty} p(1-p)^k = (1-p)^m.$$

DEFINITION 2. Let Y be a r.v. taking values in the interval $(0, 1)$. Moreover, let for all $p \in (0, 1)$ the r.v:s Y and V_p are independent. Let $N = V_Y$, that is,

$$\mathsf{P}(N \ge m) = \int_0^1 (1-y)^m d\mathsf{P}(Y < y)$$

for any $m \in \mathbb{N}$. The distribution of the r.v. N will be called *Y-mixed geometric*.

MAIN RESULTS

In the paper [8] it was shown that any gamma distribution with shape parameter no greater than one is mixed exponential. For convenience, formulate this result as the following lemma.

LEMMA 1 [8]. *The density of a gamma distribution $g(x; r, \mu)$ with $0 < r < 1$ can be represented as*

$$g(x; r, \mu) = \int_0^{\infty} ze^{-zx} p(z; r, \mu)dz,$$

where

$$p(z; r, \mu) = \frac{\mu^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbf{1}(z \ge \mu)}{(z-\mu)^r z}.$$

Moreover, a gamma distribution with shape parameter $r > 1$ cannot be represented as a mixed exponential distribution.

LEMMA 2 [12]. *For $r \in (0, 1)$ let $G_{r,\,1/2}$ and $G_{1-r,\,1/2}$ be independent gamma-distributed r.v:s. Let $\mu > 0$, $p \in (0, 1)$. Then the density $p(z; r, \mu)$ in lemma 1 corresponds to the r.v.*

$$Z_{r,\mu} = \mu(G_{r,\,1/2} + G_{1-r,\,1/2})/G_{r,\,1/2}.$$

LEMMA 3 [13]. *Let $\alpha \in (0, 1]$. Then $W_\alpha \stackrel{d}{=} W_1 \cdot S_{\alpha,1}^{-1}$ with the r.v:s on the right-hand side being independent.*

LEMMA 4. *A d. f. $F(x)$ with $F(0) = 0$ corresponds to a mixed exponential distribution if and only if the function $1 - F(x)$ is completely monotone: $F \in C_\infty$ and $(-1)^{n+1}F^{(n)}(x) \ge 0$ for all $x > 0$.*

This statement immediately follows from the Bernstein theorem [14].

THEOREM 1. *Let $\alpha \in (0, 1]$, $r \in (0, 1)$, $\mu > 0$. Then the GG-distribution with parameters r, α, μ is a mixed exponential distribution: $G^*_{r,\alpha,\mu} \stackrel{d}{=} W_1 \cdot \big(S_{\alpha,1} Z_{r,\mu}^{1/\alpha}\big)^{-1}$ with the r.v:s on the right-hand side being independent. Moreover, a GG-distribution with $\alpha r > 1$ cannot be represented as mixed exponential.*

PROOF. Prove the first assertion of the theorem. First, note that $\mathsf{P}(G_{r,\mu}^{1/\alpha} > x) = \mathsf{P}(G_{r,\mu} > x^\alpha)$. Hence, according to lemma 1 for $x \ge 0$ we have

$$\mathsf{P}(G_{r,\mu}^{1/\alpha} > x) = \mathsf{P}(W_1 > Z_{r,\mu} x^\alpha) =$$

$$= \int_0^{\infty} e^{-zx^\alpha} p(z; r, \mu)dz = \int_0^{\infty} \mathsf{P}(W_\alpha > xz^{1/\alpha}) p(z; r, \mu)dz,$$

that is, $G_{r,\mu}^{1/\alpha} \stackrel{d}{=} W_\alpha \cdot Z_{r,\mu}^{-1/\alpha}$. Now apply lemma 3 and obtain

$$G_{r,\mu}^{1/\alpha} \stackrel{d}{=} W_1 \cdot \big(S_{\alpha,1} Z_{r,\mu}^{1/\alpha}\big)^{-1}. \tag{2}$$

Second, it is easy to see that

$$G_{r,\mu}^{1/\alpha} \stackrel{d}{=} G^*_{r,\alpha,\mu} \tag{3}$$

for any $r > 0$, $\mu > 0$ and $\alpha > 0$. Now the desired assertion follows from (2) and (3).

To prove the second assertion, assume that $\alpha r > 1$ and the r.v. $G^*_{r,\alpha,\mu}$ has a mixed exponential distribution. By lemma 4 this means that the function $\psi(s) = \mathsf{P}(G^*_{r,\alpha,\mu} > s)$, $s \ge 0$, is completely monotone. But $\psi'(s) = g^*(s; r, \alpha, \mu) \ge 0$ for all $s \ge 0$, whereas $\psi''(s) = (g^*)'(s; r, \alpha, \mu) = \frac{\alpha\mu^r}{\Gamma(r)} s^{\alpha r - 2} e^{-\mu s^\alpha}\big((\alpha r - 1) - \mu\alpha s^\alpha\big) \le 0$, only if $(\alpha r - 1) - \mu\alpha s^\alpha \le 0$, that is, $s \ge s_0 \equiv \big[(\alpha r - 1)/\mu\alpha\big]^{1/\alpha} > 0$, and $\psi''(s) \ge 0$ for $s \in (0, s_0) \ne \varnothing$ contradicting the complete monotonicity of $\psi(s)$ and thus proving the second assertion. The theorem is proved.

DEFINITION 3. For $r > 0$, $\alpha \in \mathbb{R}$ and $\mu > 0$ let $\Pi_{r,\alpha,\mu}$ be a r.v. with the *GG-mixed Poisson distribution*

$$\mathsf{P}(\Pi_{r,\alpha,\mu} = k) = \frac{1}{k!}\int_0^\infty e^{-z}z^k g^*(z;r,\alpha,\mu)dz,$$

$k = 0, 1, 2...$

Since negative binomial distributions are mixed Poisson laws with gamma-mixing distributions [15], [4] the class of GG-mixed Poisson laws contains negative binomial distributions ($\alpha = 1$). Moreover, it also contains Poisson-Weibull distributions ($r = 1$) [6].

THEOREM 2. *If $r \in (0, 1]$, $\alpha \in (0, 1]$ and $\mu > 0$, then a GG-mixed Poisson distribution is a $Y_{r,\alpha,\mu}$-mixed geometric distribution*:

$$\mathsf{P}(\Pi_{r,\alpha,\mu} = k) = \int_0^1 y(1-y)^k d\mathsf{P}(Y_{r,\alpha,\mu} < y),$$

$k = 0, 1, 2...$, *where*

$$Y_{r,\alpha,\mu} \stackrel{d}{=} \frac{S_{\alpha,1}Z_{r,\mu}^{1/\alpha}}{1 + S_{\alpha,1}Z_{r,\mu}^{1/\alpha}} \stackrel{d}{=}$$

$$\stackrel{d}{=} \frac{\mu^{1/\alpha}S_{\alpha,1}(G_{r,1/2} + G_{1-r,1/2})^{1/\alpha}}{G_{r,1/2}^{1/\alpha} + \mu^{1/\alpha}S_{\alpha,1}(G_{r,1/2} + G_{1-r,1/2})^{1/\alpha}}, \quad (4)$$

where the r.v:s $S_{\alpha,1}$ and $Z_{\mu,r}$ or $S_{\alpha,1}$, $G_{r,1/2}$ and $G_{1-r,1/2}$ are independent.

PROOF. Using theorem 1 we have

$$\mathsf{P}(\Pi_{r,\alpha,\mu} = k) = -\frac{1}{k!}\int_0^\infty e^{-z}z^k d\mathsf{P}(G^*_{r,\alpha,\mu} > z) =$$

$$= -\frac{1}{k!}\int_0^\infty e^{-z}z^k d\mathsf{P}(W_1 > S_{\alpha,1}Z_{r,\mu}^{1/\alpha}z) =$$

$$= \frac{1}{k!}\int_0^\infty x\Big(\int_0^\infty e^{-z(1+x)}z^k dz\Big)d\mathsf{P}(S_{\alpha,1}Z_{r,\mu}^{1/\alpha} < x) =$$

$$= \frac{\Gamma(k+1)}{k!}\int_0^\infty \frac{x}{(1+x)^{k+1}}d\mathsf{P}(S_{\alpha,1}Z_{r,\mu}^{1/\alpha} < x) =$$

$$= \int_0^\infty \frac{x}{1+x}\Big(1 - \frac{x}{1+x}\Big)^k d\mathsf{P}(S_{\alpha,1}Z_{r,\mu}^{1/\alpha} < x).$$

Changing the variables $\frac{x}{1+x} \longmapsto y$, we finally obtain

$$\mathsf{P}(\Pi_{r,\alpha,\mu}= k) = \int_0^1 y(1-y)^k d\mathsf{P}\big(S_{\alpha,1}Z_{r,\mu}^{1/\alpha} < \tfrac{y}{1-y}\big) =$$

$$= \int_0^1 y(1-y)^k d\mathsf{P}\bigg(\frac{S_{\alpha,1}Z_{r,\mu}^{1/\alpha}}{1 + S_{\alpha,1}Z_{r,\mu}^{1/\alpha}} < y\bigg). \quad (5)$$

Moreover, (5) and lemma 2 yield representation (4). The theorem is proved.

REMARK 1. With the account of lemma 1 it is easy to verify that the density $q(y;r,\alpha,\mu)$ of the r.v. $Y_{r,\alpha,\mu}$ admits the following integral representation via the strictly stable density $f_{\alpha,1}(x)$:

$$q(y;r,\alpha,\mu) = \frac{\mu^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbf{1}(0 \le y \le 1)}{(1-y)^2} \times$$

$$\times \int_\mu^\infty f_{\alpha,1}\Big(\frac{yz^{-1/\alpha}}{1-y}\Big)\frac{dz}{(z-\mu)^r z^{1+2/\alpha}}.$$

REMARK 2. It is easily seen that the sum $G_{r,1/2} + G_{1-r,1/2}$ in (4) has the exponential distribution with parameter $\frac{1}{2}$. However, the numerator and denominator of the expression on the right-hand side of (4) are not independent.

From (4) we easily obtain the following asymptotic assertion.

COROLLARY 1. *As $\mu \to 0$, the r.v. $Y_{r,\alpha,\mu}$ is the quantity of order $\mu^{1/\alpha}$ in the sense that*

$$\mu^{-1/\alpha}Y_{r,\alpha,\mu} \Longrightarrow S_{\alpha,1}Z_{r,1}^{1/\alpha} \stackrel{d}{=}$$

$$\stackrel{d}{=} S_{\alpha,1} \cdot \Big(\frac{G_{r,1/2} + G_{1-r,1/2}}{G_{r,1/2}}\Big)^{1/\alpha},$$

where the r.v:s $S_{\alpha,1}$ and $Z_{\mu,r}$ or $S_{\alpha,1}$, $G_{r,1/2}$ and $G_{1-r,1/2}$ are independent.

Theorem 1, corollary 1, lemma 3 and theorem 1 of [12] yield the following statement.

THEOREM 3. *If $r \in (0, 1]$, $\alpha \in (0, 1]$ and $\mu > 0$, then*

$$\mu^{1/\alpha}\Pi_{r,\alpha,\mu} \Longrightarrow \frac{W_1}{S_{\alpha,1}Z_{r,1}^{1/\alpha}} \stackrel{d}{=}$$

$$\stackrel{d}{=} W_\alpha \cdot \Big(\frac{G_{r,1/2}}{G_{r,1/2} + G_{1-r,1/2}}\Big)^{1/\alpha} \stackrel{d}{=} G^*_{r,\alpha,\mu}$$

as $\mu \to 0$, where the r.v:s W_1, S_α and $Z_{r,1}$ are independent as well as the r.v:s W_α, $G_{r,1/2}$ and $G_{1-r,1/2}$.

LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN WHICH THE NUMBER OF SUMMANDS HAS THE GG-MIXED POISSON DISTRIBUTION

Consider a sequence of independent identically distributed (i.i.d.) r.v:s $X_1, X_2, \ldots$ defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Assume that $\mathsf{E}X_1 = 0$, $0 < \sigma^2 = \mathsf{D}X_1 < \infty$. For a natural $n \ge 1$ let $S_n = X_1 + \ldots + X_n$. Let $N_1, N_2, \ldots$ be a sequence of nonnegative integer random variables defined on the same probability space so that for each $n \ge 1$ the random variable N_n is independent of the sequence $X_1, X_2, \ldots$ A random sequence $N_1, N_2, \ldots$ is said to be infinitely increasing ($N_n \longrightarrow \infty$) in probability, if $\mathsf{P}(N_n \le m) \longrightarrow 0$ as $n \to \infty$ for any $m \in (0, \infty)$.

LEMMA 5. *Assume that the r.v:s $X_1, X_2, \dots$ and $N_1, N_2, \dots$ satisfy the conditions specified above and $N_n \longrightarrow \infty$ in probability as $n \to \infty$. A d.f. $F(x)$ such that*

$$\mathsf{P}\big(S_{N_n} < x\sigma\sqrt{n}\big) \Longrightarrow F(x) \quad (n \to \infty),$$

exists if and only if there exists a d.f. $Q(x)$ satisfying the conditions $Q(0) = 0$,

$$F(x) = \int_0^\infty \Phi\big(x/\sqrt{y}\big)dQ(y), \quad x \in \mathbb{R},$$

$$\mathsf{P}(N_n < nx) \Longrightarrow Q(x) \quad (n \to \infty).$$

PROOF. This lemma is a particular case of a result proved in [16], the proof of which is, in turn, based on general theorems on convergence of superpositions of independent random sequences [18]. Also see [19], theorem 3.3.2.

Re-denote $n = \mu^{-1/\alpha}$. Then $\mu = 1/n^\alpha$. Consider the r.v. $\Pi_{r,\alpha,1/n^\alpha}$. From theorem 3 it follows that $\Pi_{r,\alpha,1/n^\alpha} \to \infty$ in probability and

$$\frac{\Pi_{r,\alpha,1/n^\alpha}}{n} \Longrightarrow \frac{W_1}{S_{\alpha,1}Z_{r,1}^{1/\alpha}} \stackrel{d}{=}$$

$$\stackrel{d}{=} W_\alpha \cdot \left(\frac{G_{r,1/2}}{G_{r,1/2} + G_{1-r,1/2}}\right)^{1/\alpha} \tag{6}$$

as $n \to \infty$, where in each term the involved r.v:s are independent.

Now from (6), lemma 5 with $N_n = \Pi_{r,\alpha,1/n^\alpha}$, (1) and the well-known relation $\Lambda \stackrel{d}{=} X\sqrt{2W_1}$ we directly obtain

THEOREM 4. *Assume that the random variables $X_1, X_2, \dots$ and $N_1, N_2, \dots$ satisfy the conditions specified above. Let $r \in (0, 1]$, $\alpha \in (0, 1]$. Then*

$$\frac{S_{\Pi_{r,\alpha,1/n^\alpha}}}{\sigma\sqrt{n}} \Longrightarrow X \cdot \sqrt{G^*_{r,\alpha,\mu}} \stackrel{d}{=}$$

$$\stackrel{d}{=} X \cdot \sqrt{\frac{W_1}{S_{\alpha,1}Z_{r,1}^{1/\alpha}}} \stackrel{d}{=} \frac{\Lambda}{\sqrt{2S_{\alpha,1}Z_{r,1}^{1/\alpha}}}$$

as $n \to \infty$, where in each term the involved r.v:s are independent.

LIMIT THEOREMS FOR STATISTICS CONSTRUCTED FROM SAMPLES WITH RANDOM SIZES HAVING THE GG-MIXED POISSON DISTRIBUTIONS

Consider a sequence of i.i.d. r.v:s $X_1, X_2, \dots$ defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Let $N_1, N_2, \dots$ be a sequence of nonnegative integer random variables defined on the same probability space so that for each $n \ge 1$ the random variable N_n is independent of the sequence $X_1, X_2, \dots$ A random sequence $N_1, N_2, \dots$ is said to be infinitely increasing ($N_n \longrightarrow \infty$) in probability, if $\mathsf{P}(N_n \le m) \longrightarrow 0$ as $n \to \infty$ for any $m \in (0, \infty)$.

For $n \ge 1$ let $T_n = T_n(X_1, \dots, X_n)$ be a statistic, that is, a measurable function of the random variables $X_1, \dots, X_n$. For each $n \ge 1$ define the random variable T_{N_n} by letting $T_{N_n}(\omega) = T_{N_n(\omega)}\big(X_1(\omega), \dots, X_{N_n(\omega)}(\omega)\big)$ for every elementary outcome $\omega \in \Omega$. We will say that the statistic T_n is asymptotically normal, if there exists $\vartheta \in \mathbb{R}$ such that

$$\mathsf{P}\left(\sqrt{n}\big(T_n - \vartheta\big) < x\right) \Longrightarrow \Phi(x) \quad (n \to \infty). \tag{7}$$

LEMMA 6. *Assume that $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Let the statistic T_n be asymptotically normal in the sense of* (7). *Then a distribution function $F(x)$ such that*

$$\mathsf{P}\left(\sqrt{n}\big(T_{N_n} - \vartheta\big) < x\right) \Longrightarrow F(x) \quad (n \to \infty),$$

exists if and only if there exists a distribution function $Q(x)$ satisfying the conditions $Q(0) = 0$,

$$F(x) = \int_0^\infty \Phi\big(x\sqrt{y}\big)dQ(y), \quad x \in \mathbb{R},$$

$$\mathsf{P}(N_n < nx) \Longrightarrow Q(x) \quad (n \to \infty).$$

This lemma is a particular case of theorem 3 in [17], the proof of which is, in turn, based on general theorems on convergence of superpositions of independent random sequences [18]. Also see [19], theorem 3.3.2.

From (6), lemma 5 with $N_n = \Pi_{r,\alpha,1/n^\alpha}$ and (1) with the account of the easily verified property of GG-distributions $(G^*_{r,\alpha,\mu})^{-1} \stackrel{d}{=} G^*_{r,-\alpha,\mu}$ we directly obtain

THEOREM 5. *Let $r \in (0, 1]$, $\alpha \in (0, 1]$. Let the statistic T_n be asymptotically normal in the sense of* (7). *Then*

$$\sqrt{n}\big(T_{\Pi_{r,\alpha,1/n^\alpha}} - \vartheta\big) \Longrightarrow X \cdot \sqrt{G^*_{r,-\alpha,\mu}} \stackrel{d}{=}$$

$$\stackrel{d}{=} X \cdot \sqrt{\frac{S_{\alpha,1}Z_{r,1}^{1/\alpha}}{W_1}} \stackrel{d}{=} S_{2\alpha,0} \cdot \sqrt{\frac{Z_{r,1}^{1/\alpha}}{2W_1}}$$

as $n \to \infty$, where in each term the involved r.v:s are independent.

REMARK 3. The distribution of the limit r.v. in theorem 4 is a special case of the so-called *generalized variance gamma distributions*, see [10]. If $\alpha = 1$, then $S_{\alpha,1} \equiv 1$ and according to lemma 1 the limit law in theorem 4 turns into that of the r.v. $X\sqrt{Z_{r,1}W_1^{-1}} \stackrel{d}{=} XG_{r,1}^{-1/2}$, that is, the Student distribution with $2r$ degrees of freedom (see [20], [4]).

REMARK 4. It is worth noting that the mixing GG-distributions in the limit normal scale mixtures in theorems 4 and 5 differ only by the sign of the parameter α.

CONCLUSION

In the paper, a theorem due to L. J. Gleser stating that a gamma distribution with shape parameter no greater than one is a mixed exponential distribution was extended to generalized gamma distributions introduced by E. W. Stacy as a special family of lifetime distributions containing both gamma distributions, exponential power and Weibull distributions. It was shown that the mixing distribution is a scale mixture of strictly stable laws concentrated on the nonnegative halfline. As a corollary, the representation was obtained for the mixed Poisson distribution with the generalized gamma mixing law as a mixed geometric distribution. Limit theorems were proved establishing the convergence of the distributions of statistics constructed from samples with random sizes obeying the mixed Poisson distribution with the generalized gamma mixing law including random sums to special normal mixtures.

The obtained results can serve as a theoretical explanation of some mixed models used within the popular Bayesian approach to the statistical analysis of lifetime data related to high performance information systems.

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