RESAMPLING-APPROACH TO A TASK OF COMPARISON OF TWO RENEWAL PROCESSES

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ABSTRACT

This paper illustrates the resampling-approach to a task of comparison of two renewal processes with application to solving one inventory theory problem. The probability of the shortage absence of inventory unit is estimated. The formula for the variance calculation of estimator of interest is presented. Obtained resampling-estimators are compared with classical ones. It is shown that considered approach could be a good alternative to classical one, taking bias, variance and mean square error of estimators into account. Numerical examples illustrate the efficiency of considered method.

INTRODUCTION

Suppose we have two simple independent renewal processes $\{X_i, i=1,2,...\}$ and $\{Y_i, i=1,2,...\}$, where $\{X_i\}$ and $\{Y_i\}$ are the sequences of unnegative independent random variables, each sequence with its own common distribution (Cox 1962, Ross 1992). Let $D_m = \sum_{i=1}^{m} X_i$ and $S_m = \sum_{i=1}^{m} Y_i$ be the times of the m-th renewal for corresponding processes. The distribution functions of sequences $\{X_i\}$ and $\{Y_i\}$ are unknown, but corresponding initial samples' of sizes n_X and n_Y are available. Our purpose is the estimation of the probability $P\{D_m > S_k\}$, where $n_X \geq 2m$ and $n_Y \geq 2k$.

This problem has a lot of applications, for example, in inventory theory (Ross 1992) it occurs in the following situation. Suppose that the initial inventory level equals to K, where K is a known integer. Inventory level is increasing according to the supply and decreasing according to the demand. It is also assumed, that if the demand exceeds the supply then the shortage occurs. Our purpose is to estimate the shortage absence probability for the m-th unit's demand.

The described example can be considered in terms of renewal processes in the following way. Let the demand corresponds to the first renewal process $\{X_i, i=1,2,...\}$ and the time of the m-th renewal be the time of the m-th request of inventory unit. Let the supply corresponds to the second renewal process $\{Y_i, i=1,2,...\}$ and the time of the m-th renewal be the time of the m-th supply of inventory unit. Then the probability of interest, of the shortage absence, is the probability, that the m-th demand comes later, that the $m-K$ -th supply $D_m > S_{m-K}$. It is also assumed, that the initial inventory level K is known. We wish to investigate some properties of the different estimators of the shortage probability.

In the next section some important relationships for future purposes are given. Then follows the section, where the resampling-estimators of probability of interest are presented. The third section presents the specific cases for some distributions on the base of resampling-approach. The fourth section considers the classical estimators of the probability of interest. Further in the fifth section numerical examples illustrate the suggested approach efficiency. The last section concludes the paper.

SOME RELATIONSHIPS

Now we describe our problem more formally. Let us define the distribution functions of sequences $\{X_i\}$ and $\{Y_i\}$ as $F_1^X(x)$ and $F_1^Y(x)$, and distribution density functions as $f_1^X(x)$ and $f_1^Y(x)$. The functions $F_1^X(x)$ and $F_1^Y(x)$ are unknown, but corresponding samples $H^X = \{X_1, X_2, \ldots, X_{n_X}\}$ and $H^Y = \{Y_1, Y_2, \ldots, Y_{n_Y}\}$ are available for sequences $\{X_i\}$ and $\{Y_i\}$, where $|H^X| = n_X$ and $|H^Y| = n_Y$.

Let us define the distribution and density function of the sum of variables: $F_i^X(x)$ and $f_i^X(x)$. The sub index i in this notation means the number of addends in the sum. The mentioned above function $F_1^X(x)$ according to this notation is the distribution function of the sum presented with only one element. The upper index means the r.v. name, whose distribution function is considered.

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We are interested in the time of the *m*-th and $m-K$-th renewals:

$$D_m = \sum_{i=1}^{m} X_i \ , \ S_{m-K} = \sum_{i=1}^{m-K} Y_i \ . \quad (1)$$

Our task is to estimate the shortage absence probability $P\{D_m > S_{m-K}\}$ that the *m*-th renewal of the demand process $\{X_i\}$ comes later, than the $m-K$-th renewal of the supply process $\{Y_i\}$.

Let's consider the indicator function $\Psi(\mathbf{x},\mathbf{y})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_{m_X})$ and $\mathbf{y} = (y_1, y_2, \ldots, y_{m_Y})$ are vectors of real numbers:

$$\Psi(\mathbf{x},\mathbf{y}) = \begin{cases} 1 & if \sum_{i=1}^{m_X} x_i > \sum_{i=1}^{m_Y} y_i \\ 0 & otherwise. \end{cases} \quad (2)$$

Suppose we have two vectors of r.v. $\mathbf{X} = (X_1, X_2, \ldots X_{m_X})$ and $\mathbf{Y} = (Y_1, Y_2, \ldots Y_{m_Y})$, $m_X = m$, $m_Y = m-K$. Our purpose is to estimate the shortage absence probability $\Theta = E(\Psi(\mathbf{X},\mathbf{Y}))$. We will estimate Θ using two different approaches: classical and resampling. Classical, parametrical approach is widely known. So we consider the alternative nonparametrical resampling-approach implementation in our paper.

RESAMPLING-APPROACH

At first we consider the resampling-approach for estimation of the shortage absence probability. This method does not supposes the estimation of the distribution parameters or the construction of the empirical distribution functions to find characteristics of interest, as it is supposed by traditional methods. Alternatively we use primary data in different combinations and this fact makes possible to obtain unbiased estimators and decrease their variance. There are a lot of examples of resampling-approach successive implementation to various tasks for reliability systems, regression models, for order statistics estimation (Andronov A. and Merkuryev Yu. 2000, Wu 1986, Gentle 2000).

Resampling-approach supposes the following steps. We choose randomly m_X elements from the sample H^X and m_Y elements from the sample H^Y. The elements are taken without replacement, we remind that $n_X \geq 2m_X$, $n_Y \geq 2m_Y$. Then we calculate the corresponding value of function $\Psi(\mathbf{x},\mathbf{y})$ using formula (2). After that we return chosen elements into the corresponding samples.

We repeat this procedure during *r* realizations. Let $j_d^i(l)$, *d*=1,..,m_i be the indeces of elements from the sample H^i, $i \in \{X,Y\}$, that are chosen at the *l*-th realization. Then for the *l*-th realization we obtain the following vectors:

$$\mathbf{X}(l) = (X_{j_1^X(l)}, X_{j_2^X(l)}, \ldots, X_{j_{m_X}^X(l)}) \ ,$$

$$\mathbf{Y}(l) = (Y_{j_1^Y(l)}, Y_{j_2^Y(l)}, \ldots, Y_{j_{m_Y}^Y(l)}) \ .$$

The resampling-estimator Θ^{R*} is the arithmetical mean by *r* realizations:

$$\Theta^{R*} = \frac{1}{r}\sum_{l=1}^{r} \Psi(\mathbf{X}(l),\mathbf{Y}(l)) \ . \quad (3)$$

Obviously this estimator is unbiased:

$$E(\Theta^{R*}) = \Theta \ . \quad (4)$$

We are interested in the variance of this estimator.

Let's denote the following notations:

$$\mu = E(\Psi(\mathbf{X},\mathbf{Y})) \ , \ \mu_2 = E(\Psi(\mathbf{X},\mathbf{Y})^2) \ , \quad (5)$$

$$\mu_{11} = E\big(\Psi(\mathbf{X}(l),\mathbf{Y}(l))\Psi(\mathbf{X}(l'),\mathbf{Y}(l'))\big), \ l \neq l' \ . \quad (6)$$

Then the variance of interest can be calculated using the following formula:

$$V(\Theta^{R*}) = E(\Theta^{R*2}) - \mu^2 \ , \quad (7)$$

where

$$E(\Theta^{*2}) = \frac{1}{r}\mu_2 + \frac{r-1}{r}\mu_{11} \ . \quad (8)$$

In order to estimate the variance of the estimator, we have firstly to find the expression of the mixed moment μ_{11} from the formula (6).

To calculate the moment μ_{11} the notation of α-pairs can be used (Fioshin 2000). Let us denote $W_i(l)$, *l*=1,...,*r*, $i \in \{X,Y\}$, a subset of the sample H^i, which was used for producing the values of vectors **X**(*l*) and **Y**(*l*) correspondingly, $W_i(l) \subset H^i$. Let us denote M_i={0,1,...,m_i}, $M = M_X \times M_Y$. Let **α**=(α_X, α_Y) be an element of *M*, $\boldsymbol{\alpha} \in M$. We say that $W_i(l)$ and $W_i(l')$ produce the **α**-pair, if and only if $W_i(l)$ and $W_i(l')$ have α_i common elements: $|W_i(l) \cap W_i(l)| = \alpha_i$.

Let $A_{ll'}(\boldsymbol{\alpha})$ denote the event "subsamples (**X**(*l*),**Y**(*l*)) and (**X**(*l'*),**Y**(*l'*)) produce **α**-pair", but $P_{ll'}(\boldsymbol{\alpha})$ be the probability of this event: $P_{ll'}(\boldsymbol{\alpha}) = P\{A_{ll'}(\boldsymbol{\alpha})\}$. Because of the fact realizations *l*=1,..,*r* are statistically equivalent, we can omit the lower indices *ll'* and write P(**α**).

Let

$$\mu_{11}(\boldsymbol{\alpha}) = E\big(\Psi(\mathbf{X}(l),\mathbf{Y}(l))\Psi(\mathbf{X}(l'),\mathbf{Y}(l')) \mid A_{ll'}(\boldsymbol{\alpha})\big) \ , \quad (9)$$

then

$$\mu_{11} = \sum_{\boldsymbol{\alpha}\in M} P(\boldsymbol{\alpha})\mu_{11}(\boldsymbol{\alpha}) . \qquad (10)$$

Therefore we need to calculate $P\{\boldsymbol{\alpha}\}$ and $\mu_{11}(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha}\in M$.

The probability $P\{\boldsymbol{\alpha}\}$ can be calculated using hypergeometrical distribution**:**

$$P(\boldsymbol{\alpha}) = \prod_{i\in\{X,Y\}} \binom{m_i}{\alpha_i}\binom{n_i - m_i}{m_i - \alpha_i} \Big/ \binom{n_i}{m_i}, \qquad (11)$$

where $\binom{n}{m}$ is binomial coefficient.

Now our task is to calculate $\mu_{11}(\boldsymbol{\alpha})$, $\forall \boldsymbol{\alpha}\in M$.

Let's introduce some new notations for two different realizations *l* and *l'* of resampling-procedure. Using sums, mentioned earlier in formula (1) let's include the upper index corresponding to the realization number. Then let's divide each sum into two parts in the following way. We consider separately the common and the different elements of these sums for realizations *l* and *l'*:

$$\begin{gathered} D^{l}_{m_X} = D^{dif(ll')}_{m_X-\alpha_X} + D^{com(ll')}_{\alpha_X},\ D^{l'}_{m_X} = D^{dif(l'l)}_{m_X-\alpha_X} + D^{com(ll')}_{\alpha_X}, \\ S^{l}_{m_Y} = S^{dif(ll')}_{m_Y-\alpha_Y} + S^{com(ll')}_{\alpha_Y},\ S^{l'}_{m_Y} = S^{dif(l'l)}_{m_Y-\alpha_Y} + S^{com(ll')}_{\alpha_Y} \end{gathered} \qquad (12)$$

where

$D^{j}_{m_X} = \sum_{\xi\in W_X(j)} X_\xi$ (or $S^{j}_{m_Y} = \sum_{\xi\in W_Y(j)} Y_\xi$) is the sum of sequences $\{X_i\}$ (or $\{Y_i\}$) elements for the *j*-th realization, $j\in\{l,l'\}$,

$D^{dif(jk)}_{n}$ (or $S^{dif(jk)}_{n}$) is the sum of *n* elements from ***X***(*j*) (or ***Y***(*j*)), which are absent in ***X***(*k*) (or ***Y***(*k*)), $k, j\in\{l,l'\}, k\neq j$,

$D^{com(ll')}_{n}$ (or $S^{com(ll')}_{n}$) is the sum of *n* common elements of **X**(*l*) and ***X***(*l'*) (or ***Y***(*l*) and ***Y***(*l'*)).

Therefore we can write:

$$\begin{aligned} \mu_{11}(\boldsymbol{\alpha}) &= P\{\Psi(\mathbf{X}(l),\mathbf{Y}(l)) = 1, \Psi(\mathbf{X}(l'),\mathbf{Y}(l')) = 1 \mid \boldsymbol{\alpha}\} = \\ &= P\{D^{dif(ll')}_{m_X-\alpha_X} + D^{com(ll')}_{\alpha_X} > S^{dif(ll')}_{m_Y-\alpha_Y} + S^{com(ll')}_{\alpha_Y}, \\ &\quad D^{dif(l'l)}_{m_X-\alpha_X} + D^{com(ll')}_{\alpha_X} > S^{dif(l'l)}_{m_Y-\alpha_Y} + S^{com(ll')}_{\alpha_Y}\} = \\ &= P\{D^{dif(ll')}_{m_X-\alpha_X} + C_{\boldsymbol{\alpha}} > S^{dif(ll')}_{m_Y-\alpha_Y}, D^{dif(l'l)}_{m_X-\alpha_X} + C_{\boldsymbol{\alpha}} > S^{dif(l'l)}_{m_Y-\alpha_Y}\} = \\ &= \int_{-\infty}^{+\infty} P\{D^{dif(ll')}_{m_X-\alpha_X} + z > S^{dif(ll')}_{m_Y-\alpha_Y}, \\ &\quad D^{dif(l'l)}_{m_X-\alpha_X} + z > S^{dif(l'l)}_{m_Y-\alpha_Y}\} f^{C}(z\mid\boldsymbol{\alpha})dz, \end{aligned}$$

where $D^{com(ll')}_{\alpha_X} - S^{com(ll')}_{\alpha_Y} = C_{\boldsymbol{\alpha}}$, $f^{C}(x\mid\boldsymbol{\alpha})$ – the distribution density function of r.v. C_α. Note that for fixed value of r.v. $C_{\boldsymbol{\alpha}} = z$ the events $\{D^{dif(ll')}_{m_X-\alpha_X} + z > S^{dif(ll')}_{m_Y-\alpha_Y}\}$ and $\{D^{dif(l'l)}_{m_X-\alpha_X} + z > S^{dif(l'l)}_{m_Y-\alpha_Y}\}$ are independent. Then $\mu_{11}(\boldsymbol{\alpha})$ has the following form:

$$\begin{aligned} \mu_{11}(\boldsymbol{\alpha}) &= \int_{-\infty}^{+\infty} P\{D^{dif(ll')}_{m_X-\alpha_X} + z > S^{dif(ll')}_{m_Y-\alpha_Y}\}\cdot \\ &\cdot P\{D^{dif(l'l)}_{m_X-\alpha_X} + z > S^{dif(l'l)}_{m_Y-\alpha_Y}\} f^{C}(z\mid\boldsymbol{\alpha})dz . \end{aligned}$$

Therefore

$$\begin{aligned} R(z\mid\boldsymbol{\alpha}) &= P\{D^{dif(ll')}_{m_X-\alpha_X} + z > S^{dif(ll')}_{m_Y-\alpha_Y}\} = \\ &= P\{D^{dif(l'l)}_{m_X-\alpha_X} + z > S^{dif(l'l)}_{m_Y-\alpha_Y}\} = \\ &= \int_{-\infty}^{+\infty} F^{Y}_{m_Y-\alpha_Y}(x+z) f^{X}_{m_X-\alpha_X}(x)dx. \end{aligned} \qquad (13)$$

The r.v. $C_{\boldsymbol{\alpha}}$, has the following cumulative distribution and probability density functions:

$$\begin{aligned} F^{C}(z\mid\boldsymbol{\alpha}) &= P\{C_{\boldsymbol{\alpha}} \le z\} = P\{D^{com(ll')}_{\alpha_X} - S^{com(ll')}_{\alpha_Y} \le z\} = \\ &= P\{D^{com(ll')}_{\alpha_X} \le S^{com(ll')}_{\alpha_Y} + z\}, \\ f^{C}(z\mid\boldsymbol{\alpha}) &= \int_{-\infty}^{+\infty} f^{X}_{\alpha_X}(x+z) f^{Y}_{\alpha_Y}(x)dx. \end{aligned} \qquad (14)$$

Note that the events $\{D^{dif(ll')}_{m_X-\alpha_X} + z > S^{dif(ll')}_{m_Y-\alpha_Y}\}$ and $\{D^{dif(l'l)}_{m_X-\alpha_X} + z > S^{dif(l'l)}_{m_Y-\alpha_Y}\}$ are equal probable. Therefore we can write the formula for $\mu_{11}(\boldsymbol{\alpha})$ calculation in the following way:

$$\mu_{11}(\boldsymbol{\alpha}) = \int_{-\infty}^{+\infty} R(z\mid\boldsymbol{\alpha})^2 f^{C}(z\mid\boldsymbol{\alpha})dz. \qquad (15)$$

SPECIFIC CASES

This section includes some examples, when we illustrate the suggested approach on well known distributions.

Example: Normal distribution

Let's consider the case, when in our renewal processes r.v. *X* and *Y* of interest are normally distributed: X~N(μ_X,σ_X), Y~N(μ_Y,σ_Y). Then the sum $D^{dif(ij)}_{m_X-\alpha_X}$ from formula (12) has also normal distribution with the expectation $E(D^{dif(ij)}_{m_X-\alpha_X}) = (m_X-\alpha_X)\mu_X$ and the variance $V(D^{dif(ij)}_{m_X-\alpha_X}) = (m_X-\alpha_X)\sigma^2_X$, i.e.

$$F^{X}_{m_X-\alpha_X}(x) = \Phi\left(\frac{x-(m_X-\alpha_X)\mu_X}{\sqrt{(m_X-\alpha_X)}\sigma_X}\right), \qquad (16)$$

where $\Phi(x)$ – standard normal N(0,1) distribution function.

Analogously, the sum $S_{m_Y-\alpha_Y}^{dif(ij)}$ has also normal distribution with the expectation $(m_Y-\alpha_Y)\mu_Y$ and the variance $(m_Y-\alpha_Y)\sigma_Y^2$.

The distribution function of $C_{\boldsymbol{\alpha}}$ is also normal with the expectation $\alpha_X\mu_X-\alpha_Y\mu_Y$ and the variance $\alpha_X\sigma_X^2+\alpha_Y\sigma_Y^2$.

Then using formula (15) we can find the necessary mixed moment.

Example: Exponential distribution

Let's consider now another example, when in our renewal preceses r.v. X and Y of interest have exponential distribution with parameters λ and ν correspondingly. Then the distribution function $F_{m_X-\alpha_X}^X(x)$ of the sum $D_{m_X-\alpha_X}^{dif(ij)}$ from formula (12) has Erlang distribution with parameters λ and $m_X-\alpha_X$. The distribution function $F_{m_Y-\alpha_Y}^Y(x)$ of the other sum $S_{m_Y-\alpha_Y}^{dif(ij)}$ from formula (12) has also Erlang distribution with parameters ν and $m_Y-\alpha_Y$. We also define with letter G (with corresponding indices*)* an additional function to corresponding distribution function: $G(x)=1-F(x)$.

Let us consider the integral from (13) in the following way:

$$R(c\,|\,\boldsymbol{\alpha})=1-\int_{-\infty}^{+\infty}G_{m_Y-\alpha_Y}^Y(y+c)f_{m_X-\alpha_X}^X(y)dy=$$

$$=1-\int_{-\infty}^{\max(0,-c)}T(y\,|\,\boldsymbol{\alpha})dy-\int_{\max(0,-c)}^{+\infty}T(y\,|\,\boldsymbol{\alpha})dy,$$

where $T(y|\boldsymbol{\alpha})$ is the underintegral expression.

All intermediate proofs and calculus for those two integral parts are given in the Appendix.
Finally, we have:

$$R(c\,|\,\boldsymbol{\alpha})=1-F_{m_X-\alpha_X}^X(\max(0,-c))-$$
$$-e^{-\nu c}\lambda^{m_X-\alpha_X}\sum_{i=0}^{m_Y-\alpha_Y-1}\frac{\nu^i}{i!}\sum_{p=0}^{i}\binom{i}{p}c^p\frac{1}{(\lambda+\nu)^{m_X-\alpha_X-p+i}}\cdot \qquad (17)$$
$$\cdot\frac{(i-p+m_X-\alpha_X-1)!}{(m_X-\alpha_X-1)!}\cdot G_X(\max(0,-c)\,|\,\boldsymbol{\alpha}),$$

where $G_X(x\,|\,\boldsymbol{\alpha})$ - additional function for Erlang distribution function with parameters $\lambda+\nu$, $m_X-\alpha_X+i-p$.

Now consider the probability density function from (14) for the r.v. $D_{\alpha_X}^{com(ij)}-S_{\alpha_Y}^{com(ij)}=C_{\boldsymbol{\alpha}}$, where $D_{\alpha_X}^{com(ij)}$ and $S_{\alpha_Y}^{com(ij)}$ have Erlang distribution with parameters (α_X, λ) and (α_Y, ν) correspondingly.

Then

$$f^C(c\,|\,\boldsymbol{\alpha})=\frac{\lambda^{\alpha_X}\nu^{\alpha_Y}e^{-\lambda c}}{(\alpha_X-1)!(\alpha_Y-1)!}\cdot$$
$$\cdot\sum_{p=0}^{\alpha_X-1}\binom{\alpha_X-1}{p}\cdot c^p\frac{(\alpha_X+\alpha_Y-p-2)!}{(\lambda+\nu)^{\alpha_X+\alpha_Y-p-1}}G_X(\max(0,-c)), \qquad (18)$$

where $G_X(x)$ corresponds to additional function for Erlang distribution with parameters $\lambda+\nu$ and $\alpha_X+\alpha_Y-p-1$. The proof for formula (18) are given in the Appendix.

Then using formula (15) we can find the necessary mixed moment.

Note that with some modifications those formulas are also available for Erlang distribution.

CLASSICAL APPROACH

Classical approach to the estimation of the probability of interest is a parametrical one. It supposes the point estimation of the parameters of the distribution, if we know the distribution type of the initial samples H^i, $i=\{X, Y\}$. We intend to estimate the parameters of the known types of distributions.

Example: Exponential distribution

Let's consider an example, when r.v. X and Y have exponential distribution with parameters λ and ν correspondingly. The sum of exponentially distributed r.v. has Erlang distribution. The probability of interest $\Theta=P\{D_{m_X}>S_{m_Y}\}$, notation from formula (1), now is:

$$\Theta=\int_0^{+\infty}G_{m_X}^X(y)f_{m_Y}^Y(y)dy=$$
$$=\int_0^{+\infty}e^{-\lambda y}\sum_{i=1}^{m_X}\frac{(\lambda y)^{i-1}}{(i-1)!}\nu\frac{(\nu y)^{m_Y-1}}{(m_Y-1)!}e^{-\nu y}dy= \qquad (19)$$
$$=\sum_{i=0}^{m_X-1}\frac{\nu^{m_Y}}{(\lambda+\nu)^{m_Y+i}}\frac{\lambda^i}{i!}\prod_{p=0}^{i-1}(m_Y+p).$$

The classical approach supposes using the point estimators instead of the values of λ and ν: $\lambda^*=n_X/D_{n_X}$, $\nu^*=n_Y/S_{n_X}$. It gives the estimator:

$$\Theta^{C*}=\sum_{i=0}^{m_X-1}\frac{\nu^{*m_Y}}{(\lambda^*+\nu^*)^{m_Y+i}}\frac{\lambda^{*i}}{i!}\prod_{p=0}^{i-1}(m_Y+p), \qquad (20)$$

where $\prod_{p=0}^{-1}=1$.

Now we are able to calculate the expectation and the variance of Θ^{C*}.

Example: Normal distribution

Now let's consider the case, where *X* and *Y* have normal distribution, correspondingly $N(\mu_X,\sigma_X)$ and $N(\mu_Y,\sigma_Y)$. The real probability of the shortage absence in this case can be calculated as follows:

$$\Theta = P\{D_{m_X} > S_{m_Y}\} = P\{D_{m_X} - S_{m_Y} > 0\}.$$

It is known, that the distribution of this difference also has normal distribution. Therefore,

$$\Theta(\boldsymbol{\mu},\boldsymbol{\sigma}) = 1 - \Phi\left(\frac{0-(m_X\mu_X - m_Y\mu_Y)}{\sqrt{m_X\sigma_X^2 + m_Y\sigma_y^2}}\right).$$

If we try to estimate this probability, the classical approach supposes the estimation of the parameters $\boldsymbol{\mu} = (\mu_X,\mu_Y)$, $\boldsymbol{\sigma} = (\sigma_X,\sigma_Y)$ using available sample populations. We have the estimator:

$$\Theta^{C*} = \Theta^*(\boldsymbol{\mu}^*,\boldsymbol{\sigma}^*) = 1 - \Phi\left(\frac{0-(m_X\mu_X^* - m_Y\mu_Y^*)}{\sqrt{m_X\sigma_X^{*2} + m_Y\sigma_y^{*2}}}\right)$$

Then the expectation and the second moment of this estimator have the following expressions:

$$E(\Theta^{C*}) =$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta^*(\boldsymbol{\mu}^*,\boldsymbol{\sigma}^*) f_{\mu_{XY}}(\mu_{XY}^*) f_{\sigma_X}(\sigma_X^*) f_{\sigma_Y}(\sigma_Y^*) d\mu_{XY}^* d\sigma_X^* d\sigma_Y^*,$$
$$E(\Theta^{C*2}) =$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta^*(\boldsymbol{\mu}^*,\boldsymbol{\sigma}^*)^2 f_{\mu_{XY}}(\mu_{XY}^*) f_{\sigma_X}(\sigma_X^*) f_{\sigma_Y}(\sigma_Y^*) d\mu_{XY}^* d\sigma_X^* d\sigma_Y^*,$$

where distribution density function $f_{\mu_{XY}}(x)$ of the $m_X\mu_X^* - m_Y\mu_Y^*$ has normal distribution with parameters $m_X\mu_X - m_Y\mu_Y$ and $\sqrt{\sigma_X{}^2 m_X^2/n_X + \sigma_Y{}^2 m_Y^2/n_Y}$, and distribution density function $f_{\sigma_i}(x)$ of the $n_i\sigma_i^{*2}/\sigma_i^2$ has "chi-square" distribution with $n_i - 1$ degrees of freedom, *i*={*X*,*Y*}. Then the expression for variance of Θ^{C*} is: $V(\Theta^{C*}) = E(\Theta^{C*2}) - E(\Theta^{C*})^2$ and for the mean square error $ER(\Theta^{C*}) = V(\Theta^{C*}) + (\Theta - E(\Theta^{C*}))^2$.

NUMERICAL RESULTS

Consider the case when r.v. *X* and *Y* have normal distribution with parameters $\mu_X = \mu_Y = 2$, $\sigma_X = \sigma_Y = 1$. The real estimators of probabilities of the shortage absence are presented in the Table 1.

Let our sample sizes be equal $n = n_X = n_Y$, and we consider the *m*-th unit's demand and different initial inventory levels *K*=0..3. All calculations have performed for *r* = 1000 realizations.

We intend to compare the variance of estimators of resampling-approach with the mean square error of classical approach. It is so because of resampling-approach estimators are unbiased, but classical ones on the contrary have bias.

Table 1 Real probabilities of shortage absence Θ

	Θ			
	K=0	*K*=1	*K*=2	*K*=3
n=6, *m*=3	.5	.814	---	---
n=10, *m*=5	.5	.748	.921	.988
n=10, m=4	.5	.775	.949	---
n=12, *m*=6	.5	.727	.897	.977

Table 2 Experimental results for Classical Θ^{C*} and Resampling Θ^{R*} estimators

		K=0	*K*=1	*K*=2	*K*=3
n=6 *m*=3	$V(\Theta^{C*})$	.067	.034	---	---
	$B(\Theta^{C*})$	0	.022	---	---
	$ER(\Theta^{C*})$	.067	.035	---	---
	$V(\Theta^{R*})$	.112	.047	---	---
n=10 *m*=5	$V(\Theta^{C*})$	.061	.043	.015	.002
	$B(\Theta^{C*})$	0	.028	.029	.013
	$ER(\Theta^{C*})$	.061	.044	.015	.002
	$V(\Theta^{R*})$	.087	.055	.014	.001
n=10 *m*=4	$V(\Theta^{C*})$	.053	.032	.006	---
	$B(\Theta^{C*})$	0	.021	.018	---
	$ER(\Theta^{C*})$	.053	.033	.007	---
	$V(\Theta^{R*})$	.069	.039	.005	---
n=12 *m*=6	$V(\Theta^{C*})$	.06	.045	.019	.004
	$B(\Theta^{C*})$	0	.028	.033	.019
	$ER(\Theta^{C*})$	.06	.046	.02	.004
	$V(\Theta^{R*})$	.085	.058	.02	.002

In the Table 2 and at the fig. 1 we can see the resampling-estimators' variance $V(\Theta^{R*})$ comparing with classical approach estimators' variance $V(\Theta^{C*})$, bias $B(\Theta^{C*})$, and mean square error $ER(\Theta^{C*})$. The table shows how changes the results depending on different sample sizes *n*, unit's number *m* and initial inventory level *K*.

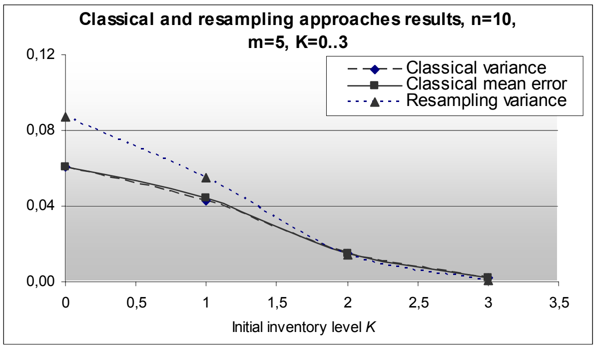

Figure 1 Classical and resampling-approach results

Analyzing table's results we can draw the conclusion that the variance and corresponding mean square error of both approaches decreases with the increasing of sample sizes *n*, *m*, and initial inventory level *K*. The variance of resampling-estimators is almost always near the traditional one. However resampling-estimators are unbiased. Taking as the criterion the mean square error resampling gives even better results for big values of *K*.

CONCLUSION

Resampling-approach can be successfully used for obtaining the estimators of parameters of interest of the renewal processes. Obtained formulas allow calculating the variance of the estimators for resampling and classical approaches. Numerical examples show the efficiency of suggested approach, taking estimators' mean square error as efficiency criterion. This approach can be good alternative to traditional one.

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APPENDIX

Below the proof for formula (17) is given:

$$R(c\,|\,\boldsymbol{\alpha}) = 1 - \int_{-\infty}^{+\infty} G^{Y}_{m_Y-\alpha_Y}(y+c) f^{X}_{m_X-\alpha_X}(y)dy =$$

$$= 1 - \int_{-\infty}^{\max(0,-c)} T(y\,|\,\boldsymbol{\alpha})dy - \int_{\max(0,-c)}^{+\infty} T(y\,|\,\boldsymbol{\alpha})dy.$$

Let us consider the parts of this integral separately:

$$\int_{-\infty}^{\max(0,-c)} T(y\,|\,\boldsymbol{\alpha})dy = \int_{-\infty}^{\max(0,-c)} G^{Y}_{m_Y-\alpha_Y}(y+c) f^{X}_{m_X-\alpha_X}(y)dy =$$

$$= \int_{0}^{\max(0,-c)} 1\cdot f^{X}_{m_X-\alpha_X}(y)dy = F^{X}_{m_X-\alpha_X}(\max(0,-c)).$$

Let us for future calculations define $m_X - \alpha_X = a$, $m_Y - \alpha_Y = b$:

$$\int_{\max(0,-c)}^{+\infty} T(y\,|\,\boldsymbol{\alpha})dy = \int_{\max(0,-c)}^{\infty} G^{Y}_{b}(y+c) f^{X}_{a}(y)dy =$$

$$= \int_{\max(0,-c)}^{+\infty} e^{-\nu(y+c)} \sum_{i=0}^{b-1} \frac{\nu^i (y+c)^i}{i!} \lambda \frac{(\lambda y)^{a-1}}{(a-1)!} e^{-\lambda y} dy =$$

$$= \frac{e^{-\nu c}\lambda^a}{(a-1)!} \sum_{i=0}^{b-1} \frac{\nu^i}{i!} \int_{\max(0,-c)}^{+\infty} e^{-(\lambda+\nu)y} \sum_{p=0}^{i} \binom{i}{p} y^{i-p+a-1} c^p dy =$$

$$= \frac{e^{-\nu c}\lambda^a}{(a-1)!} \sum_{i=0}^{b-1} \frac{\nu^i}{i!} \sum_{p=0}^{i} \binom{i}{p} c^p \int_{\max(0,-c)}^{+\infty} e^{-(\lambda+\nu)y} y^{i-p+a-1} dy =$$

$$= \frac{e^{-\nu c}\lambda^a}{(a-1)!} \sum_{i=0}^{b-1} \frac{\nu^i}{i!} \sum_{p=0}^{i} \binom{i}{p} c^p \frac{(i-p+a-1)!}{(\lambda+\nu)^{a-p+i}} \cdot$$

$$\cdot \int_{\max(0,-c)}^{+\infty} e^{-(\lambda+\nu)y} \frac{y^{i-p+a-1}(\lambda+\nu)^{a-p+i}}{(i-p+a-1)!} dy.$$

Below the proof for formula (18) is given:

$$f_C(c\,|\,\boldsymbol{\alpha}) = \int_{-\infty}^{+\infty} f^{X}_{\alpha_X}(y+c) f^{Y}_{\alpha_Y}(y)dy =$$

$$= \int_{\max(0,-c)}^{+\infty} f^{X}_{\alpha_X}(y+c) f^{Y}_{\alpha_Y}(y)dy =$$

$$= \int_{\max(0,-c)}^{+\infty} \frac{\lambda^{\alpha_X}(y+c)^{\alpha_X-1}}{(\alpha_X-1)!} e^{-\lambda(y+c)} \frac{\nu(\nu y)^{\alpha_Y-1}}{(\alpha_Y-1)!} e^{-\nu y} dy =$$

$$= \frac{e^{-\lambda c}\lambda^{\alpha_X}\nu^{\alpha_Y}}{(\alpha_X-1)!(\alpha_Y-1)!} \sum_{p=0}^{\alpha_X-1} \binom{\alpha_X-1}{p} c^p \frac{(\alpha_X+\alpha_Y-p-2)!}{(\lambda+\nu)^{\alpha_X+\alpha_Y-p-1}} \cdot$$

$$\cdot \int_{\max(0,-c)}^{+\infty} \frac{y^{\alpha_Y+\alpha_X-p-2}(\lambda+\nu)^{\alpha_X+\alpha_Y-p-1}}{(\alpha_X+\alpha_Y-p-2)!} e^{-(\lambda+\nu)y} dy.$$